Fields with pseudo-exponentiation

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1 Introduction

This research is motivated by the study of model-theoretical properties of classical 'analytic' structures, i.e. ones having natural analytic representation (see also [Z]). For example, the structure of complex numbers as a field with exponentiation

$$\mathbf{C}_{\exp} = (\mathbf{C}, +, \cdot, \exp).$$

One of the questions we can ask is whether $\mathbf{C}_{\exp}$ is **quasi-minimal,** i.e. any definable subset of $\mathbf{C}_{\exp}$ is either countable or of power continuum. Another question is about homogeneity of the structure; we do not know any its automorphism except the identity and the complex conjugation. In general we would like to understand the nature of analytic dimension in a context close to model-theoretic stability theory.

A slightly weaker analytic structure $\mathbf{C}^{(2)}_{\exp}$ is a two-sorted structure with both sorts $\mathbf{C}(1)$ and $\mathbf{C}(2)$ being copies of complex numbers, on both sorts the field structure is given and there is a mapping $\exp : \mathbf{C}(1) \to \mathbf{C}(2)$ in the language. The model theory of the both structures, as well as of many others of this kind, seems very hard to study directly.

We study here model theory of abstract analogues of $\mathbf{C}^{(2)}_{\exp}$. We start by considering the class $\mathcal{E}_p$ of two-sorted structures of the form (D, ex, R), D, the domain of a mapping ex is a field of characteristic zero, R, a field of characteristic p and ex is a homomorphism of the additive group of D onto the multiplicative group $R^\times$ of R. Following the ideology of Hrushovski's construction of non-classical structures with nice dimension notion (see [H]),

we consider a notion of a *predimension* $\delta(X)$ for finite subsets $X \subseteq D$ defined as

$$\delta(X) = \text{tr.d.}(X) + \text{tr.d.}(\text{ex}(X)) - \dim_{\mathbb{Q}}(X),$$

here $\dim_{\mathbb{Q}}(X)$ is the dimension of the $\mathbf{Q}$-linear space generated by X.
We then define the subclass $\mathcal{S}_p$ of $\mathcal{E}_p$, the *weak Schanuel class* as the class of structures where $\delta(X) \geq 0$ for all X and the kernel of ex is *standard* (a cyclic additive subgroup in case $p = 0$). Recall that the Schanuel conjecture [L] states that, given $\mathbf{Q}$-linearly independent complex numbers $x_1, \ldots, x_n$, the inequality

$$\text{tr.d.}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) - n \geq 0$$

holds. We prove that the weak Schanuel class $\mathcal{S}_p$ is non-empty.
We study the predimension δ and the relation $A \leq B$ following the Hrushovski pattern. We define a notion of *exponentially-algebraically closed structures*, which are just existentially closed structures with respect to the relation $A \leq B$ in class $\mathcal{S}_p$. Again, by standard model theory as used by Hrushovski, in the class $\mathcal{SC}_p$ of exponentially-algebraically closed structures δ gives rise to the *dimension notion* ∂.
On the other hand class $\mathcal{SC}_p$ is shown to be axiomatizable inside $\mathcal{S}_p$. More precisely, it is axiomatized by an axiom stating that any *normal* system of exponential-algebraic equations has a solution in the structure. The definition of a normal system of exponential-algebraic equations is explicit and purely algebraic. It follows
A necessary and sufficient condition for the structure of complex numbers with exp *to be in* $\mathcal{SC}_0$ *is to satisfy the weak Schanuel conjecture and to have solutions to any normal system of exponential-algebraic equations.*
There are grounds to believe that complex numbers do satisfy the axiom.

Next we proceed by developing analogues of atomic and prime structures in class $\mathcal{SC}_p$. A notion of quasi-atomic types and structures as well as quasi-prime structures over a given subset are introduced in style of [Sh]. We restrict the study to the case $p = 0$. Then the usual nice properties for the notions hold. Given a subset $C \leq \mathbf{D}$ in a structure in $\mathbf{D} \in \mathcal{SC}_0$ we construct a quasi-prime structure $E(C) \leq \mathbf{D}$. If the kernel of ex, the additive subgroup $K = \{x \in D : \text{ex}(x) = 1\}$, is compact in the profinite topology then $E(C)$ is determined uniquely up to isomorphism over C. Now take C a ∂-independent set of power λ. We call $E(C)$ in this case *the canonical weak*

Schanuel structures of power λ (canonical for short). This notion may be useful because $\mathbf{C}^{(2)}_{\exp}$ looks very much like a canonical structure of power $2^{\aleph_0}$ with the *standard* kernel isomorphic to $\mathbf{Z}$.
This is certainly hard to prove, since the proof would include a proof of the weak Schanuel conjecture. Still it would be interesting to study the properties of canonical structures compairing these to the known properties of $\mathbf{C}$. It follows from what was stated above that
For $\lambda > \aleph_0$, *a canonical structure with the standard kernel is quasi-minimal, i.e. any definable subset of* E_λ *is either of power not greater than* $\aleph_0$ *or of power* λ.

In a canonical structure with standard kernel E_λ ($\lambda > \aleph_0$) we study the notion of dimension for definable subsets $S \subseteq E^n$, denoted adim (S), which we call **the pseudo-analytic dimension**. We define it analogously to the definition of Morley Rank:

$$\text{adim } S = \max_{\bar{x} \in S} \partial \bar{x}.$$

The pseudo-analytic dimension is a good analogue of the analytic dimension in $\mathbf{C}$:
(i) adim $S \geq 0$ *for any non-empty* S;
(ii) for non-empty S adim $S = 0$ *iff* card $(S) \leq \aleph_0$;
(iii) for non-empty S adim $S \geq m > 0$ *iff* S *can be projected to* D^m *so that the complement to the projection in* D^m *is of pseudo-analytic dimension less than* m*;*
(iv) For any irreducible algebraic varieties $V \subseteq D^n$ *and* $W \subseteq R^n$
$V \cap \ln W \neq \emptyset$ *implies* adim $(V \cap \ln W) \geq \dim V + \dim W - n$.

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2 Definitions and notation

For technical reasons it is more suitable to work with the equivalent class of one sorted structures on D in the language $\mathcal{L}_p$ which is the version of extended language of fields. On D the following operations and relations are defined: the addition $'+'$ and multiplication by all rational scalars; n-ary predicates $V(x_1, \ldots, x_n)$ for each algebraic subvariety $V \subseteq D^n$ defined and irreducible over 0 an equivalence relation E, this allows to consider $R = D/E$, we think of R as of a field; n-ary predicates $EW(y_1, \ldots, y_n)$ for each algebraic subvariety $W \subseteq R^n$ definable and irreducible over 0.

The interpretation assumes $E(x, y) \equiv [\mathrm{ex}(x) = \mathrm{ex}(y)]$,
$V(x_1, \ldots, x_n) \equiv [\langle x_1, \ldots, x_n \rangle \in V]$,
$EW(x_1, \ldots, x_n) \equiv [\langle \mathrm{ex}(x_1), \ldots, \mathrm{ex}(x_n) \rangle \in W]$.

Definition $\mathcal{E}_p$ is the class of structures $\mathbf{D}$ in language $\mathcal{L}_p$ such that D is a field of characteristic zero, E an equivalence relation on D which is congruent with respect to the relations $EW(x_1, \ldots, x_n)$, D/E when considered with relations $EW(x_1, \ldots, x_n)$ is embeddable in the group $R^\times$ where R is a field of characteristic p so that the canonical mapping

$$\mathrm{ex} : D \to R^\times$$

is a homomorphism of the additive group of D into the multiplicative group $R^\times$.

Notation For the sake of simplicity we fix p below and write $\mathcal{E}$ instead of $\mathcal{E}_p$.

For finite $X, X' \subseteq D,\ \ Y, Y' \subseteq R$
tr.d.(X) the the transcendence degree of X over 0;
$\dim_{\mathbb{Q}}(X)$ the dimension of the vector space $\mathrm{span}_{\mathbf{Q}}(X)$ generated by X over $\mathbf{Q}$;
tr.d.(Y) the transcendence degree of Y over 0;
$\delta(X)$ **the predimension** of finite $X \subseteq D$:
$\delta(X) = \mathrm{tr.d.}(X) + \mathrm{tr.d.}(\mathrm{ex}(X)) - \dim_{\mathbb{Q}}(X)$;
$\delta(X/X') = \delta(XX') - \delta(X')$, and $XX' = X \cup X'$;

for infinite $A \subseteq D$ $\delta(X/A) \geq k$, $k \in \mathbf{Z}$, means that for any finite $Y \subseteq A$ there is finite $Y \subseteq Y' \subseteq A$ such that $\delta(X/Y') \geq k$, and $\delta(X/A) = k$ means $\delta(X/A) \geq k$ and not $\delta(X/A) = k+1$.

$\mathrm{tr.d.}(X/X') = \mathrm{tr.d.}(XX') - \mathrm{tr.d.}(X')$;
$\mathrm{tr.d.}(Y/Y') = \mathrm{tr.d.}(YY') - \mathrm{tr.d.}(Y')$;
$\dim_{\mathrm{Q}}(X/X') = \dim_{\mathrm{Q}}(XX') - \dim_{\mathrm{Q}}(X')$;
ker is the name of a unary predicate of type EW $x \in \ker \equiv \mathrm{ex}(x) = 1$.

Denote $\mathcal{S}$ the subclass of $\mathcal{E}$ consisting of all D satisfying the condition:

$$\delta(X) \geq 0 \text{ for all finite } X \subseteq D.$$

Denote $sub\mathcal{E}$ the class of structures A in language $\mathcal{L}_p$ satisfying :
A is an additive divisible subgroup of the additive group of a field D of characteristic zero and A/E can be embedded into $R^\times$ for a field R of characteristic p and the canonical mapping $\mathrm{ex}_A : A \to R^\times$ is a homomorphism.

Denote $sub\mathcal{S}$ the subclass of $sub\mathcal{E}$ consisting of A which satisfy $\delta(X) \geq 0$ for any finite $X \subseteq A$.

For W an algebraic variety, $\bar{b} = \langle b_1, \ldots, b_l \rangle$ denote

$$W(\bar{b}) = \{\langle x_{l+1}, \ldots, x_{n+l}\rangle : \langle b_1, \ldots, b_l, x_{l+1}, \ldots, x_{n+l}\rangle \in W\}.$$

Lemma 2.1 *If* $X = \{x_1, \ldots x_{n+l}\} \subseteq D$, $X' = \{x_1, \ldots x_l\}$, $\bar{x} = \langle x_1, \ldots x_{n+l}\rangle$, $\bar{x}' = \langle x_1, \ldots x_l\rangle$ *then:* $\mathrm{tr.d.}(X) = \dim V$, *where* $V \subseteq D^{n+l}$ *is the minimal algebraic variety over* 0 *containing* $\bar{x}$;
$\dim_{\mathrm{Q}}(X) = \dim L$ *where* $\bar{L}$ *is the minimal linear subspace of* D^{n+l} *containing* $\bar{x}$ *and given by homogeneous linear equations over* $\mathbf{Q}$;
$\mathrm{tr.d.}(\mathrm{ex}X)$ *is the dimension of the minimal subvariety of* R^{n+l} *over* 0 *containing* $\mathrm{ex}(\bar{x})$;
$\dim_{\mathrm{Q}}(X/X') = \dim L(0^l)$, *where* 0^l *is a string of* l *zeroes.*

Proof Immediate from definitions.□

Lemma 2.2 *(i)* $\delta(X/X') = \text{tr.d.}(X/X') + \text{tr.d.}(\text{ex}X/\text{ex}X') - \dim_{\mathrm{Q}}(X/X')$;
(ii) For any $A \subseteq D$ *and finite* $X \subseteq D$ *there is a finite* $Y \subseteq A$*, such that, if* $Y \subseteq Y' \subseteq A$*, then* $\delta(X/Y') = \delta(X/A)$.

Proof (i) is immediate from definitions.
(ii) follows from (i) if we choose $Y \subseteq A$, such that $\text{tr.d.}(X/Y) = \text{tr.d.}(X/A)$, $\text{tr.d.}(\text{ex}(X)/\text{ex}(Y)) = \text{tr.d.}(\text{ex}(X)/\text{ex}(A))$, $\dim_{\mathrm{Q}}(X/Y) = \dim_{\mathrm{Q}}(X/A)$. The choice is possible, since $\text{tr.d.}(X/Y)$, $\text{tr.d.}(\text{ex}(X)/\text{ex}(Y))$ and $\dim_{\mathrm{Q}}(X/Y)$ are non-increasing functions of Y. $\Box$

Remark The condition $\dim_{\mathrm{Q}}(X/A) = \dim_{\mathrm{Q}}(X/Y)$ for $Y \subseteq A$ is satisfied iff $\text{span}_Q(X) \cap A \subseteq Y$. Correspondingly with the transcendense degree.

Notation For $A, B \in sub\mathcal{E}$ denote $A \leq B$ the fact that $A \subseteq B$ as structures and $\delta(X/A) \geq 0$ for all finite $X \subseteq B$.

Lemma 2.3 *For any structure* A *of the class* $sub\mathcal{E}$ *and finite* $X, Y, Z \subseteq A$:
(i) If $\text{span}_Q(X') = \text{span}_Q(X)$ *then* $\delta(X') = \delta(X)$.
(ii) If $\text{span}_Q(X'Y) = \text{span}_Q(XY)$ *then* $\delta(X/Y) = \delta(X'/Y)$.
(ii) If $\text{span}_Q(Y) = \text{span}_Q(Y')$ *then* $\delta(X/Y) = \delta(X/Y')$.
(iv) $\delta(XY/Z) = \delta(X/YZ) + \delta(Y/Z)$.

Proof Immediate from the definitions.$\Box$

Lemma 2.4 *For* $A, B, C \in sub\mathcal{E}$
(i) if $A \leq B$ *and* $B \leq C$*, then* $A \leq C$;
(ii) if $A \leq B$, $Y \subseteq B$, $\delta(Y/A) = 0$*, then* $AY \leq B$.

Proof (i) Let $X \subseteq C$ finite and $Z \subseteq A$ large enough finite so that $\delta(X/Z) = \delta(X/A)$. We need to prove that $\delta(X/Z) \geq 0$. Choose $Y \subseteq B$ finite so that $\text{span}_Q(YZ) = A \cap \text{span}_Q(XZ)$. Then $\dim_{\mathrm{Q}}(X/YZ) = \dim_{\mathrm{Q}}(X/B)$. From the definition of δ it follows that $\delta(X/YZ) \geq \delta(X/B) \geq 0$. Hence $\delta(XY/Z) = \delta(X/YZ) + \delta(Y/Z) \geq 0$. At last notice that $\delta(X/Z) = \delta(XY/Z)$ by Lemma 2.3 (ii).

(ii) For any $X \subseteq B$ we want to show $\delta(X/AY) \geq 0$. It is given by

$$\delta(X/AY) = \delta(XY/A) - \delta(Y/A) = \delta(XY/A) \geq 0,$$

since $A \leq B$. $\square$

Notation Let $A \in sub\mathcal{E}$ and $X \subseteq A$ finite. Denote

$$\partial_A(X) = \min\{\delta(X') : X \subseteq X' \subseteq A\}.$$

Lemma 2.5 *Let $A \in sub\mathcal{E}$ and $X \subseteq A$ finite. Choose $X' \subseteq A$ finite such that*

$$\delta(X') = \partial_A(X).$$

Then $X' \leq A$.

Proof Immediately from the definition.$\square$

Lemma 2.6 *Let $A, B \in sub\mathcal{E}$, $A \leq B$ and X a finite subset of A. Then*

$$\partial_A(X) = \partial_B(X).$$

Proof Let a finite $Y \subseteq B$ be such that

$$\delta(XY) = \partial_B(X).$$

Let Y_0 be a **Q**-linear basis of Y over A and $X_1 \subseteq A$ a finite superset of X such that

$$\mathrm{span}_Q(X_1) = \mathrm{span}_Q(XY) \cap A.$$

Then $\dim_{\mathrm{Q}}(Y_0/A) = \dim_{\mathrm{Q}}(Y_0/X_1)$, $\mathrm{tr.d.}(Y_0/A) \leq \mathrm{tr.d.}(Y_0/X_1)$, and $\mathrm{tr.d.}(\mathrm{ex}Y_0/\mathrm{ex}A) \leq \mathrm{tr.d.}(\mathrm{ex}Y_0/\mathrm{ex}X_1)$. It follows

$$\delta(Y_0/X_1) \geq \delta(Y_0/A) \geq 0.$$

Also

$$\mathrm{span}_Q(XY) = \mathrm{span}_Q(X_1Y_0).$$

Hence

$$\delta(XY) = \delta(X_1Y_0) = \delta(X_1) + \delta(Y_0/X_1).$$

By the above proved $\delta(XY) \geq \delta(X_1)$ and by the definition $\delta(X_1) \geq \partial_A(X)$. Thus $\partial_B(X) \geq \partial_A(X)$, and the converse holds by the definition.$\Box$

Lemma 2.7 *Suppose $A \in sub\mathcal{S}$, $A' \in sub\mathcal{E}$, $A' = \mathrm{span}_Q(AX)$, and $\delta(X'/A) \geq 0$ for all finite $X' \subseteq \mathrm{span}_Q X$. Then $A' \in sub\mathcal{S}$ and $A \leq A'$.*

Proof We may assume that X is **Q**-linearly independent over A. Let $Z \subseteq A'$, $Z = \{z_1, \ldots z_n\}$, and $z_i = x_i + y_i$ for some $x_i \in \mathrm{span}_Q(X)$, $y_i \in A$. Let $\{x_1, \ldots x_k\}$ be a **Q**-linear basis of $\{x_1, \ldots x_n\}$. Then, using Lemma 2.3,
$\delta(Z) = \delta(x_1 + y_1, \ldots x_k + y_k, y'_{k+1}, \ldots, y'_n)$ for $y'_{k+1}, \ldots, y'_n$ appropriate **Q**-linear combinations of $y_1, \ldots y_n$.
Rewrite

$$\delta(Z) = \delta(\{x_1 + y_1, \ldots x_k + y_k\}/\{y'_{k+1}, \ldots, y'_n\}) + \delta(y'_{k+1}, \ldots, y'_n).$$

By the assumtions $\delta(y'_{k+1}, \ldots, y'_n) \geq 0$. On the other hand
$\delta(\{x_1 + y_1, \ldots x_k + y_k\}/\{y'_{k+1}, \ldots, y'_n\}) \geq \delta(\{x_1, \ldots x_k\}/A) \geq 0$
since
$\mathrm{tr.d.}(\{x_1 + y_1, \ldots x_k + y_k\}/\{y'_{k+1}, \ldots, y'_n\}) \geq \mathrm{tr.d.}(\{x_1 + y_1, \ldots x_k + y_k\}/A) \geq \mathrm{tr.d.}(\{x_1, \ldots x_k\}/A)$,
$\mathrm{tr.d.}(\mathrm{ex}\{x_1 + y_1, \ldots x_k + y_k\}/\mathrm{ex}\{y'_{k+1}, \ldots, y'_n\}) \geq$
$\mathrm{tr.d.}(\mathrm{ex}\{x_1 + y_1, \ldots x_k + y_k\}/\mathrm{ex}A) \geq \mathrm{tr.d.}(\mathrm{ex}\{x_1, \ldots x_k\}/\mathrm{ex}A)$
and
$\dim_{\mathrm{Q}}(\{x_1+y_1, \ldots x_k+y_k\}/\{y'_{k+1}, \ldots, y'_n\}) = k = \dim_{\mathrm{Q}}(\{x_1, \ldots x_k\}/A)$. Thus

$$\delta(Z) \geq 0.$$

The same argument shows that

$$\delta(Z/A) \geq 0.$$

$\Box$

Definition Define for $p > 0$

$$\mathbf{Z}[\frac{1}{p}] = \{\frac{m}{p^n} : m \in \mathbf{Z}, n \in N\}.$$

Let $A \in sub\mathcal{E}$. A is said to be **with standard kernel** if

$$\mathrm{Ker} = \ker_{|A} = \{a \in A : \mathrm{ex}(a) = 1\} = \begin{cases} \pi \cdot \mathbf{Z} & \text{if } p = 0 \\ \pi \cdot \mathbf{Z}[\frac{1}{p}] & \text{if } p > 0 \end{cases}$$

for some transcendental $\pi \in D$.
A is said to be **with compact kernel** if $\ker_{|A}$ is an algebraically compact additive group.
Denote

$$\hat{\mathbf{Z}}_{(p)} = \prod_{l \neq p,\ l \text{ prime number}} Z_l,$$

the direct products of additive groups of l-adic integers. These groups are algebraically compact (see [F], Ch. 7).
A is said to be **with a full kernel** if for $\ker = \ker_{|A} = \{a \in A : \mathrm{ex}(a) = 1\}$ the group $A/\ker$ is isomorphic to the multiplicative subgroup of an algebraically closed field containing all the torsion points.

Proposition 1 *For any characteristic p*
(i) there is an $A \in sub\mathcal{S}_p$ with standard full kernel;
(ii) there is an $A \in sub\mathcal{S}_p$ with compact full kernel.

Proof (i) Take an additive subgroup $A = \pi \cdot \mathbf{Q} \subseteq D$ for $\mathbf{Q}$ the prime subfield of D and π a transcendental element in D. Define $H = A/\mathrm{Ker}$ for Ker the standard kernel with generator π defined above. Then H considered as multiplicative group is characterized by the property that it is a torsion group such that any equations of the form $x^n = h$ has for any h exactly n solutions when n is prime to p and only one when $n = p^k$. In other words H is isomorphic to the torsion subgroup of an algebraically closed field R of characteristic p. Define ex as the canonical homomorphism $A \to R^\times$ corresponding to this isomorphism. Since π is transcendental, $\delta(X) = 0$ for any finite $X \subseteq A$.
(ii) In a large field D of characteristic 0 consider an additive divisible subgroup A of power continuum and such that the generators of A as a vector space over $\mathbf{Q}$ are algebraically independent. Such a group is isomorphic to to the divisible hull of $\hat{\mathbf{Z}}(p)$ and any algebraic dependence between elements of A is a linear dependence over $\mathbf{Q}$. Fix $K \subseteq A$ the corresponding subgroup isomorphic to $\hat{\mathbf{Z}}(p)$. Define $H = A/K$. Then H is a torsion group such that

any equations of the form $x^n = h$ has for any h exactly n solutions when n is prime to p and only one when $n = p^k$, i.e. H is isomorphic to the torsion subgroup of an algebraically closed field R of characteristic p. Define ex as the canonical homomorphism $A \to R^\times$ corresponding to this isomorphism. Since the generators of A are algebraically independent, $\mathrm{tr.d.}(X) = \dim_{\mathbb{Q}}(X)$ for any finite $X \subseteq A$. Thus $\delta(X) = 0$ for any finite $X \subseteq A$. $\Box$

Lemma 2.8 *Suppose $A \in sub\mathcal{S}$ and A is with full kernel. Then there is $D \in \mathcal{S}$ and an embedding of A into D such that $A \leq D$ and* $\ker_{|D} = \ker_{|A}$.

Proof Choose a field D of characteristic zero and an algebraically closed field R such that $A \subseteq D$, $A/E \subseteq R^\times$ and $\mathrm{tr.d.}(D/A) = \mathrm{tr.d.}(R/\mathrm{ex}A) \geq \aleph_0$. We want to define $\mathrm{ex} : D \to R^\times$ extending ex_A so that $\mathbf{D} \in \mathcal{S}$.
Denote $A_0 = A$, $\mathrm{ex}_0 = \mathrm{ex}_A$ and $H_0 = \mathrm{ex}_0(A_0)$.
Proceed by induction defining A_α, H_α and an endomorphism

$$\mathrm{ex}_\alpha : A_\alpha \to H_\alpha$$

by choosing
on the even steps: the first element $a \in D \setminus A_\alpha$ and define $\mathrm{ex}_{\alpha+1}(a)$ to be any element in $R^\times \setminus acl(H_\alpha)$. Put $A_{\alpha+1} = A_\alpha + \mathbf{Q} \cdot a$ and extend $\mathrm{ex}_{\alpha+1}$ to $A_{\alpha+1}$ as a group homomorphism. Put $H_{\alpha+1} = \mathrm{ex}_{\alpha+1}(A_{\alpha+1})$.
on the odd steps: the first element $h \in R^\times \setminus H_\alpha$ and define a to be any element in $D \setminus \mathrm{acl}(A_\alpha)$ and $\mathrm{ex}_{\alpha+1}(a) = h$. Put $A_{\alpha+1} = A_\alpha + \mathbf{Q} \cdot a$ and extend $\mathrm{ex}_{\alpha+1}$ to $A_{\alpha+1}$ as a group homomorphism. Put $H_{\alpha+1} = \mathrm{ex}_{\alpha+1}(A_{\alpha+1})$.
On both even and odd steps it follows from Lemma 2.7 that $A_{\alpha+1} \in sub\mathcal{S}$ and $A_\alpha \leq A_{\alpha+1}$.
Also,

$$\ker_{|A_{\alpha+1}} = \ker_{|A_\alpha}$$

since if $\mathrm{ex}(qa + a') = 1$ for some rational $q = \frac{m}{n}$ and $a' \in A_\alpha$ then $h^m = g^n$ for $h = \mathrm{ex}(a)$, $g = \mathrm{ex}(a') \in H_\alpha$. Since A_α is with full kernel, H_α contains a root of degree m of g^n,and $h \notin H_\alpha$ only $q = 0$ is possible. $\Box$

Notation Let K be a full kernel for some $\mathbf{D}$ in $\mathcal{S}$. Denote $\mathcal{S}(K)$ the class of structures with kernel K.

3 Exponentially-algebraically closed structures of $\mathcal{S}(K)$

In this section we consider the class of structures with a given full kernel K. We extend the language $\mathcal{L}_p$ by naming all elements of K, thus when we say that an algebraic variety V is defined over some $C \subseteq D$, we mean the parameters in the definition of V are from the field generated by $C \cup K$. We start by giving some basic definitions and notations.

Definition For $C \subseteq D$ and algebraic varieties $V \subseteq D^n$, $W \subseteq R^n$ we say the **pair** (V, W) **is definable over** C meaning V is definable over C and W over $\mathrm{ex}(span_Q(C))$. If the varieties are irreducible over the corresponding sets, then the pair is said to be irreducible over C.

V is said to be **free of additive dependencies over** C if no $\bar{a} \in V$ generic over C satisfies $m_1 \cdot a_1 + \ldots + m_n \cdot a_n = c$ for some $c \in \mathrm{span}(C \cup K)$ and non-zero tuple of integers $m_1, \ldots, m_n$. V is said to be **aboslutely free** of additive dependencies over C if it is free over $\mathrm{acl}(C \cup K)$. W is said to be **free of multiplicative dependencies over** C if no $\bar{b} \in W$ generic over $\mathrm{ex}(C)$ satisfies $a_1^{m_1} \cdot \ldots \cdot a_n^{m_n} = r$ for some $r \in \mathrm{ex}(\mathrm{span}(C))$. W is said to be **absolutely free** of multiplicative dependencies if it is free over $\mathrm{acl}(\mathrm{ex}(C))$. A pair (V, W) is said to be **a free (absolutely free) pair** if both V and W are free (absolutely free) of additive and multiplicative dependencies correspondingly.

Let $W \subseteq R^n$ be an algebraic variety defined and irreducible over some $\mathrm{ex}(C)$ for some $C = span_Q(C \cup K) \subseteq D$. With any such W we associate a sequence $\{W^{\frac{1}{l}} : l \in \mathbf{N}\}$ of algebraic varieties which are definable and irreducible over $\mathrm{ex}(C)$ and satisfy the following:
$W^1 = W$ and for any $l, m \in \mathbf{N}$ the mapping

$$[m] : \langle y_1, \ldots y_n \rangle \mapsto \langle y_1^m, \ldots y_n^m \rangle$$

maps $W^{\frac{1}{lm}}$ onto $W^{\frac{1}{l}}$. Such a sequence is said to be **the sequence associated with** W over C.
Also with any $\langle y_1, \ldots y_n \rangle \in W$ as above we associate a sequence

$$\{\langle y_1, \ldots y_n \rangle^{\frac{1}{l}} : l \in \mathbf{N}\}$$

such that for any $l, m \in \mathbf{N}$ the mapping

$$[m] : \langle y_1, \ldots y_n \rangle \mapsto \langle y_1^m, \ldots y_n^m \rangle$$

maps $\langle y_1, \ldots y_n \rangle^{\frac{1}{lm}}$ onto $\langle y_1, \ldots y_n \rangle^{\frac{1}{l}}$. Such a sequence is said to be **associated** with $\langle y_1, \ldots y_n \rangle$.

Let (V, W), (V', W') be pairs over C, (V, W) irreducible, $V' \subsetneqq V$, $W' \subsetneqq W$, $\{W^{\frac{1}{l}} : l \in \mathbf{N}\}$ a sequence associated with W. Then the triple

$$\tau = (V \setminus V', W \setminus W', \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$$

is said to be **an almost finite [quantifier-free] type over** C. A pair of the form $(V \setminus V', W \setminus W')$ is said to be a **finite type** over C.
A tuple $\langle a_1, \ldots a_n \rangle \in D^n$ is said to be realising type τ if $\langle a_1, \ldots a_n \rangle \in V \setminus V'$, $\mathrm{ex}\langle a_1, \ldots a_n \rangle \in W \setminus W'$ and $\mathrm{ex}\frac{1}{l}\langle a_1, \ldots a_n \rangle \in W^{\frac{1}{l}}$ for all $l \in \mathbf{N}$.

A pair (V, W) is said to be a **normal pair over** C if in some extensions of the fields there are $\langle a_1, \ldots, a_n \rangle \in V$ and $\langle b_1, \ldots, b_n \rangle \in W$ such that for any $k \leq n$ independent integer vectors $m_i = \langle m_{i,1}, \ldots m_{i,n} \rangle$, $i = 1, \ldots, k$, and

$$a_i' = m_{i,1}a_1 + \ldots + m_{i,n}a_n, \quad b_i' = b_1^{m_{i,1}} \cdot \ldots \cdot b_n^{m_{i,n}}$$

it holds

$$\mathrm{tr.d.}(\langle a_1', \ldots, a_k' \rangle / C) + \mathrm{tr.d.}(\langle b_1', \ldots, b_k' \rangle / \mathrm{ex}(C)) \geq k.$$

Equivalently, for the varieties

$$V'_{1,\ldots,k} = \mathrm{locus}_C(a_1', \ldots, a_k') \text{ and } W'_{1,\ldots,k} = \mathrm{locus}_{\mathrm{ex}(C)}(b_1', \ldots, b_k')$$

it must hold

$$\dim V'_{1,\ldots,k} + \dim W'_{1,\ldots,k} \geq k.$$

Notice that in case

$$\langle a_1', \ldots, a_k' \rangle = \langle a_{i_1}, \ldots, a_{i_k} \rangle \text{ and } \langle b_1', \ldots, b_k' \rangle = \langle b_{i_1}, \ldots, b_{i_k} \rangle$$

the varieties $V'_{1,\ldots,k}$ and $W'_{1,\ldots,k}$ are just the projections $V_{i_1,\ldots,i_k}$ and $W_{i_1,\ldots,i_k}$ of the initial varieties on the $(i_1, \ldots, i_k)$-subspaces.

Remark Given a non-degenerate integer matrix $M = \{m_i : i = 1, \ldots, m_n\}$, we consider the mapping

$$[M] : \bar{a}\bar{b} \mapsto \bar{a}'\bar{b}'$$

determined by the n integer vectors of the matrix as in the definition above. This is a linear transformation on D^n and a rational transformation on R^n. We will call the mapping a **linearly induced mapping** $[M]$.

Definition A structure $\mathbf{D}$ in $\mathcal{S}(K)$ is said to be **$\mathcal{S}(K)$-exponentially-algebraically closed** (e.a.c.) if for any $\mathbf{D}' \in \mathcal{S}(K)$, such that $\mathbf{D} \leq \mathbf{D}'$, any almost finite quantifier-free type over D which is realised in $\mathbf{D}'$ has a realisation in $\mathbf{D}$.

The class of $\mathcal{S}(K)$-exponentially-algebraically closed structures is denoted $\mathcal{SC}(K)$. We omit K when the kernel is fixed.

We continue by studying properties of associated sequences. A sequence associated with $\bar{y}$ is not uniquely determined; for $\bar{y}^{\frac{1}{l}}$ there are l^n possible values. Evidentely, one can get all the values multiplying a value $\bar{y}^{\frac{1}{l}}$ by all $\bar{\xi} = \langle \xi_1, \ldots, \xi_n \rangle$, where ξ_i's are roots of unity of degree l. It follows

Lemma 3.1 *Let $\bar{y} \in W$ and $\{\bar{y}^{\frac{1}{l}} \in l \in \mathbf{N}\}$, $\{W^{\frac{1}{l}} : l \in \mathbf{N}\}$ be sequences associated with $\bar{y}$ and W correspondingly. Then there is a sequence $\{\bar{\xi}(l) : l \in \mathbf{N}\}$ of roots of unity of power l, such that*

$$\bar{\xi}(l) \cdot \bar{y}^{\frac{1}{l}} \in W^{\frac{1}{l}}$$

for all $l \in \mathbf{N}$ and $\xi(l \cdot m)^m = \xi(l)$ for all $m \in \mathbf{N}$.

Lemma 3.2 *Assume $\mathbf{D} \in sub\mathcal{S}$ is with a compact kernel, W a nonempty algebraic subvariety of R^n and $\{W^{\frac{1}{l}} : l \in \mathbf{N}\}$ a sequence associated with W. Then there is $\bar{x} \in D^n$ such that*

$$\mathrm{ex}(\frac{1}{l} \cdot \bar{x}) \in W^{\frac{1}{l}}$$

for all $l \in \mathbf{N}$.

Proof Take any $\bar{y} \in W$ and $\bar{v} \in D^n$ such that $\mathrm{ex}(\bar{v}) = \bar{y}$. Evidentely, $\{\mathrm{ex}(\frac{1}{l} \cdot \bar{v}) : l \in \mathbf{N}\}$ is a sequence associated with $\bar{y}$. By Lemma 3.1 there is

a sequence $\{\bar{\xi}(l) : l \in \mathbf{N}\}$ such that $\bar{\xi}(l) \cdot \mathrm{ex}(\frac{1}{l} \cdot \bar{v}) \in W^{\frac{1}{l}}$ for all $l \in \mathbf{N}$. Let $\bar{\alpha}(l) \in D^n$ be such that $\mathrm{ex}(\bar{\alpha}(l)) = \bar{\xi}(l)$ for all $l \in \mathbf{N}$. Consider a system of equations in the additive language in variables $\bar{\chi}$, $\bar{z}(l)$ $(l \in \mathbf{N})$ running in $\ker^n$:

$$\{\bar{\chi} = l \cdot \bar{\alpha}(l) + l \cdot \bar{z}(l) : l \in \mathbf{N}\}.$$

Any finite subsystem

$$\{\bar{\chi} = l \cdot \bar{\alpha}(l) + l \cdot \bar{z}(l) : l \leq m\}$$

of the system has a solution $\bar{\chi} = m!\bar{\alpha}(m!)$. Indeed,

$$\mathrm{ex}(\frac{1}{l} \cdot \bar{\chi}) = \bar{\xi}(m!)^{\frac{m!}{l}} = \bar{\xi}(l) = \mathrm{ex}(\alpha(l)),$$

hence $\bar{\chi} - l \cdot \bar{\alpha}(l) \in l \cdot \ker$. Thus the whole system has a solution $\bar{\chi} \in \ker^n$. Put $\bar{x} = \bar{\chi} + \bar{v}$.$\square$

If the number of the varieties $W^{\frac{1}{l}}$ for a given W grows with l, then by the symmetry coming from the action of Galois group (multiplication by roots of unity of degree l), it follows that the number of the sequences associated with W is $2^{\aleph_0}$. This is the case when W is not free of multiplicative dependence. Indeed, if $y_1^{m_{i,1}} \cdot \ldots \cdot y_1^{m_{i,n}} = c$ holds on W then for $W^{\frac{1}{l}}$ holds $y^{m_{i,1}} \cdot \ldots \cdot y^{m_{i,n}} = c^{\frac{1}{l}}$ with exactly l different choices for $c^{\frac{1}{l}}$. Nevertheless, in case W is free of multiplicative dependencies F.Voloch answered our question by proving

Proposition 2 *[F.Voloch] Let W be an irredicible algebraic variety such that W is not contained in a coset of any proper torus. Then there exist a number c such that the number of irreducible components of $W^{\frac{1}{l}}$ is bounded by c independently on l.*

Proof First we reduce the proof to the case when W is a curve by cutting W by a generic linear subspace of appropriate dimension. Denote $K = R(W)$ the field of rational functions on the curve W defined over R. Let $x_1, \ldots, x_n \in K^\times$ be the coordinate functions, which are multiplicatively independent over $R^\times$ by assumptions.
Claim. Let H be the group generated by $x_1, \ldots, x_n$ in K^*. Then there is

$d>0$, such that $\#H/H\cap(K^*)^l \geq dl^n$ for any $l\geq 1$.
Indeed, by hypothesis $x_1,\dots,x_n$ generate a free abelian group of rank n in K^*/R^* isomorphic to H, and so $\#H/H^l = l^n$. So we need to show that $\#(H\cap K^l/H^l)$ is bounded independent of l. Let $\mathcal{D}$ be the group of divisors of K/R and $\mathcal{D}_S$ the subgroup generated by $S=\{v|\exists i\ \ v(x_i)\neq 0\}$. So $\mathcal{D}_S$ is a f.g. abelian group and H embeds in $\mathcal{D}_S$. Let

$$G=\mathcal{D}_S\cap(H\otimes\mathbf{Q})\subseteq\mathcal{D}_S\otimes\mathbf{Q}$$

which is also a lattice and $H\subseteq G$ with G/H finite, $\#G/H=c$. Now, if $x\in H\cap(K^*)^l$ then $x=y^l$, $y\in K^*$. But $(y)\in\mathcal{D}_S$ and also $H\otimes\mathbf{Q}$, so $(y)\in G$, so $y^c\in H$, so $x^c=y^{cl}\in H^l$, hence $H\cap K^l/H^l$ is an abelian group of exponent c generated by $\leq n$ elements, so $\#H\cap K^l/H^l\leq c^n$ and $d=1/c^n$. Claim proved.
Let $W^{\frac{1}{l}}$ be an irreducible curve such that

$$[l](W^{\frac{1}{l}})=W.$$

Then for the field $R(W^{\frac{1}{l}})$ of rational functions of the curve by Galois theory

$$|R(W^{\frac{1}{l}}):K|=\#H/H\cap(K^*)^l.$$

The latter is bounded from below by the estimate dl^n, by the claim. On the other hand the mapping $[l]$ on the full inverse $[l]^{-1}(W)$ by Bezout theorem is of degree l^n. It follows that the number of irreducible components in the inverse image is at most $1/d=c$. □

Corollary 1 *If W is an algebraic variety absolutely free of multiplicative dependencies then any irreducible component of W satisfies assumptions of Proposition 2 and thus there is an $l_0\in\mathbf{N}$ such that sequences $\{W^{\frac{1}{l}}:l\in\mathbf{N}\}$ and $\{\bar{y}^{\frac{1}{l}}:l\in\mathbf{N}\}$ satisfy*

$$\bar{y}^{\frac{1}{l}}\in W^{\frac{1}{l}}\ \textit{for all}\ \ l\in\mathbf{N}\ \textit{iff}\ \bar{y}^{\frac{1}{l}}\in W^{\frac{1}{l}}\ \textit{for all}\ l\leq l_0.$$

In other words, the sequence $\{W^{\frac{1}{i}}:i\in\mathbf{N}\}$ is determined by its cut of length l_0.

Lemma 3.3 *Let $W \subseteq R^{n+m}$ be an algebraic variety definable and irreducible over some subfield F and absolutely free of multiplicative dependencies. Let A be a torsion-free divisible subgroup of the additive group of D of rank m, $\bar{a} \in A^m$, $W_a = W(\mathrm{ex}(\bar{a}))$. Then over $\mathrm{ex}(A) \cdot F$ any associated sequence $\{W_a^{\frac{1}{l}} : l \in \mathbf{N}\}$ is determined by its finite cut.*

Proof We may assume $\bar{a}$ generates A over $\mathbf{Q}$.
Let $\{\bar{b}^{\frac{1}{l}} : l \in \mathbf{N}\}$ be a realisation of an associated sequence $\{W_a^{\frac{1}{l}} : l \in \mathbf{N}\}$. Then the algebraic locus of $\bar{b}^{\frac{1}{l}}$ for an $l \in \mathbf{N}$ over $\mathrm{ex}(A)$ is of the form $W'(\mathrm{ex}(\bar{a}'))$ for some $\bar{a}' \in A^m$, $W' \subseteq R^{n+m}$ over F. Since $\bar{a}$ generates A, for some $k \in \mathbf{N}$ $\mathrm{ex}(\bar{a}')$ is a rational combination of $\mathrm{ex}(\frac{1}{lk}\bar{a})$. Hence we may assume $\bar{a}' = \frac{1}{lk}\bar{a}$ in the expression above. On the other hand the locus of $(n+m)$-tuple $\bar{b}^{\frac{1}{lk}}\mathrm{ex}(\frac{1}{lk}\bar{a})$ over F is given by a member $W^{\frac{1}{lk}}$ of a sequence associated with W. In other words $W'(\mathrm{ex}(\bar{a}'))$ can be represented as $(W^{\frac{1}{lk}}(\mathrm{ex}(\frac{1}{lk}\bar{a}))^k$ for some $k \in N$. Suppose $l \geq l_0$. Then $W^{\frac{1}{lk}}$ is determined uniquely once $W^{\frac{1}{l}}$ and k is given. It follows

$$(W^{\frac{1}{lk}}(\mathrm{ex}(\frac{1}{lk}\bar{a}))^k = W^{\frac{1}{l}}(\mathrm{ex}(\frac{1}{l}\bar{a}).$$

Finally notice that the sequence $\{W^{\frac{1}{l}}(\mathrm{ex}(\frac{1}{l}\bar{a}) : l \in \mathbf{N}\}$ is determined by its l_0-cut. $\square$

Notation For subsets $A \subseteq B \subseteq D$ with $\mathbf{D} \in \mathcal{S}(K)$ the notion $A \leq B$ is applied in the same sense as for substructures.

Now we come to study the rank notion ∂_D for $\mathbf{D}$ e.a.c..

Lemma 3.4 *For $\mathbf{D} \in \mathcal{SC}$, given finite $A \subseteq D$ and $\mathbf{D}' \in \mathcal{S}(K)$ such that $D \leq D'$,*

$$\partial_D(A) = \partial_{D'}(A).$$

Proof Immediate from the definitions.$\square$

By Lemma above we may omit D when writing ∂_D.

Lemma 3.5 *For any finite $A \subseteq D$ and any $a, b \in D$*
(i) $\partial(A) \leq \partial(aA) \leq \partial(A) + 1$;
(ii) $\partial(abA) = \partial(aA) = \partial(A)$ *implies* $\partial(bA) = \partial(A)$;
(iii) $\partial(abA) = \partial(aA) \& \partial(A) < \partial(bA)$ *implies* $\partial(abA) = \partial(bA)$;
(iv) $\partial(aA) = \partial(A) = \partial(bA)$ *implies* $\partial(abA) = \partial(A)$;
(v) $\partial(aA) = \partial(A)$ *implies* $\partial(bA) = \partial(abA)$.

Proof (i) follows immediately from the definitions of δ and d. (ii) and (iii) are immediate from (i).
(iv) Let $B' \supseteq aA$, $B'' \supseteq bA$ be such that $\delta(B') = \partial(aA)$ and $\delta(B'') = \partial(bA)$. Denote $B = B' \cap B''$.
Notice that $\delta(B' \cup B'') \leq \delta(B'')$. Indeed by Lemma 2.3(iii)
$\delta(B' \cup B'') = \delta(B'/B'') + \delta(B'') =$
$= [\mathrm{tr.d.}(B'/B'') + \mathrm{tr.d.}(\mathrm{ex}(B')/\mathrm{ex}(B'')) - \dim_{\mathbb{Q}}(B'/B'')] + \delta(B'')$.
By the modularity of linear dimension $\dim_{\mathbb{Q}}(B'/B'') = \dim_{\mathbb{Q}}(B'/B)$. Also, by properties of algebraic dependence
$\mathrm{tr.d.}(B'/B'') \leq \mathrm{tr.d.}(B'/B)$ and $\mathrm{tr.d.}(\mathrm{ex}(B')/\mathrm{ex}(B'')) \leq \mathrm{tr.d.}(\mathrm{ex}(B')/\mathrm{ex}(B))$
Hence $\delta(B'/B'') \leq \delta(B'/B)$. The latter is less or equal to zero by the choice of A, B' and B.
Now, since $abA \subseteq B' \cup B''$ and $\delta(B' \cup B'') \leq \delta(B'') = \partial(A)$, we have $\partial(abA) = \partial(A)$.
(v) is immediate from (iv).$\square$

Corollary 2 *The operator*

$$A \mapsto \mathrm{cl}(A) = \{b : \partial(Ab) = \partial(A)\}$$

in $\mathbf{D}$ *is a closure operator.* $\mathrm{cl}(A)$ *is said to be the ∂-closure of A.*

Definition Let $C \subseteq D$ and $\bar{a} \in D^n$ be given. The ex**-locus of** $\bar{a}$ **over** C is given as

$$(V, W, \{W^{\frac{1}{l}} : l \in \mathbf{N}\}),$$

where $V \subseteq D^n$ is the minimal algebraic variety over $C \cup K$ containing $\bar{a}$ and $W^{\frac{1}{l}} \subseteq R^n$ the minimal algebraic variety over $\mathrm{ex}(\mathrm{span}(C))$ containing $\mathrm{ex}(\frac{1}{l}\bar{a})$. We will call (V, W) the [incomplete] ex-locus.

Lemma 3.6 *Let $C, A \in sub\mathcal{S}$, $C \leq A$, and let $\bar{a}$ be a linearly independent over C string of elements of A. Then the incomplete* ex*-locus (V, W) of $\bar{a}$ over C is normal, V is free of additive dependencies, and absolutely free if C is an algebraically closed subfield of D. If C is with full kernel and* $\mathrm{ex}(C)$ *is a subfield of R, then W is free of multiplicative dependencies. If* $\mathrm{ex}(C)$ *is algebraically closed, then W is absolutely free.*

Proof Let $\langle b_1, \dots, b_n \rangle$ in the definition of normality be $\mathrm{ex}(\bar{a})$. Then the inequalities in the definition of normality follow from $C \leq A$.
An additive dependence for V would mean by definition of V a linear dependence of $\bar{a}$ over C, which does not hold.
A multiplicative dependence for W is equivalent to $\mathrm{ex}(\bar{a})$ being multiplicatively dependent over the subfield generated by $\mathrm{ex}(C)$, which is equivalent under the assumptions to $\bar{a}$ being linearly dependent over C. $\Box$

Lemma 3.7 *Suppose $\mathbf{D} \in \mathcal{SC}$, $\mathbf{D} \leq \mathbf{D}'$. Let $\bar{a}$ be a finite string in D' $\tau = (V \setminus V', W \setminus W', \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$ and $(V, W, \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$ the ex-locus of $\bar{a}$ over D. Then there is a realisation of τ in $\mathbf{D}$.*

Proof We may assume $\bar{a}$ is linearly independent over D. Then by Lemma 3.6 (V, W) is normal and absolutely free pair, since D and R are algebraically closed fields. By Corollary 1 τ is equivalent to a finite type. By the assumption for $\mathbf{D}$, τ is realised in $\mathbf{D}$.$\Box$

Lemma 3.8 *Let $K \subseteq C \subseteq D$, $\tau = (V \setminus V', W \setminus W', \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$ be an almost finite type over C and assume the pair (V, W) is normal over C, V is free of additive dependencies and W is absolutely free of multiplicative dependencies over C. Then there is $\bar{a}$ in $\mathbf{D}$ realising τ. Moreover in some extension $\mathbf{D}' \geq \mathbf{D}$ $\bar{a}$ can be chosen generic in V (of maximal tr.d.) over C and* $\mathrm{ex}(\bar{a})$ *generic in W over* $\mathrm{ex}(C)$.

Proof Let $\mathbf{D}'$ be a normal extension of $\mathbf{D}$ with D' and R' of infinite transcendence degree. Applying a linearly induced transformation we may assume that for any $\bar{a} = \langle a_1, \dots, a_n \rangle \in V$ in D', generic over C, $a_1, \dots, a_k$ are linearly

independent over $\mathrm{acl}(C)$ and $a_{k+1}, \ldots, a_n \in \mathrm{acl}(C)$. And since V is free over C $a_1, \ldots, a_n$ are linearly independent over C.
Denote $W_{k+1,\ldots,n}$ the variety induced by W on $\{k+1, \ldots, n\}$-coordinates. It follows from normality that $\dim W_{k+1,\ldots,n} = n-k$ and thus by letting $b_i^{\frac{1}{l}} = \mathrm{ex}(\frac{1}{l}a_i)$ for each $k < i \leq n$ and each $l \in \mathbf{N}$ we get $\langle b_{k+1}, \ldots, b_n\rangle^{\frac{1}{l}} \in W_{k+1,\ldots,n}^{\frac{1}{l}}$ generically. Now extend $\langle b_{k+1}, \ldots, b_n\rangle^{\frac{1}{l}}$ to $\{\langle b_1, \ldots, b_n\rangle^{\frac{1}{l}} \in W^{\frac{1}{l}}$ for each $l \in \mathbf{N}$.
By absolute freeness, $b_1, \ldots, b_n$ are multiplicatively independent over $\mathrm{acl}(\mathrm{ex}(C))$.

Extend ex to $A = D + \mathrm{span}(a_1, \ldots a_n) = D + \mathrm{span}(a_1, \ldots, a_k)$ as:

$$\mathrm{ex}(\sum_{i \leq k} \frac{m_i}{l} a_i + d) = \prod_{i \leq k} (b_i^{\frac{1}{l}})^{m_i} \cdot \mathrm{ex}(d)$$

for any integers m_i, $l \neq 0$ and element $d \in D$. The definition is consistent since $a_1, \ldots, a_k$ are linearly independent over $\mathrm{acl}(C)$ and hence over D. So the formula defines a homomorphism. The kernel of the homomorphism coincides with that of ex on D, since $\prod_{i \leq k} (b_i^{\frac{1}{l}})^{m_i} \cdot r = 1$ for some m_i and $r \in R$ implies by genericity $r \in \mathrm{acl}(\mathrm{ex}(C))$ and hence by the assumptions $m_1 = \ldots = m_k = 0$ and $r = 1$. Thus $A \in sub\mathcal{E}$ and is with full kernel. Notice that by the normality for any independent integer vectors $m_i = \langle m_{i,1}, \ldots m_{i,n}\rangle$, $i = 1, \ldots, k$, it holds $\delta(m_1\bar{a}, \ldots, m_k\bar{a}/D) \geq 0$.
Thus $D \subseteq A$ satisfy the assumptions of Lemma 2.7 and hence $A \in sub\mathcal{S}$, $D \leq A$. By the choice $\bar{a}$ realises τ. Since $\mathbf{D} \in \mathcal{SC}$ by Lemma 3.7 there is a realisation of the type in $\mathbf{D}$. □

Lemma 3.9 *Let $K \subseteq C \subseteq D$, and assume the pair (V, W) is free and normal over C. Then for any $V' \subsetneq V$ and $W' \subsetneq W$ over C there is $\bar{a} \in V \setminus V'$ living in $\mathbf{D}$ such that $\mathrm{ex}(\bar{a}) \in W \setminus W'$. Moreover in some extension $\mathbf{D'} \geq \mathbf{D}$ $\bar{a}$ can be chosen generic in V over C and $\mathrm{ex}(\bar{a})$ generic in W over $\mathrm{ex}(C)$.*
If the kernel is compact then for any sequence $\{W^{\frac{1}{l}} : l \in \mathbf{N}\}$ we can find a realisation $\bar{a}$ of type $(V \setminus V', W \setminus W', \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$.

Proof We may again assume that for any generic $\bar{a} \in V$ in some extension $a_1, \ldots, a_k$ are linearly independent over $\mathrm{acl}(C)$ and $a_{k+1}, \ldots, a_n \in \mathrm{acl}(C) \subseteq$

D. Also by transformations we may assume that for any generic $\bar{b}$ $b_1, \ldots, b_s$ are multiplicatively independent over R and $b_{s+1}, \ldots, b_k \in \mathrm{acl}(\mathrm{ex}(C)) \subseteq R$ for some $s \leq k$. It follows now from normality that the algebraic type of $a_{s+1}, \ldots, a_k$ is characterised by the property that the elements are algebraically independent over C.
Denote $W_{s+1,\ldots,k}$ the variety induced by W on $\{s+1, \ldots, k\}$-coordinates. Now, choose first a generic $\langle b_{s+1}, \ldots, b_k \rangle \in W_{s+1,\ldots,k}$ over $\mathrm{ex}(C)$. By the above noted $b_{s+1}, \ldots, b_k \in R$. In case the kernel is compact consider also an associated sequence

$$\{\langle b_{s+1}, \ldots, b_k \rangle^{\frac{1}{l}} \in W^{\frac{1}{l}}_{s+1,\ldots,k} : l \in \mathbf{N}\}.$$

Now choose for $s < i \leq k$ $a_i \in D$ so that $\mathrm{ex}(a_i) = b_i$ and, in the case of compact kernel, by Lemma 3.2, $\mathrm{ex}(\frac{1}{l}a_i) = b_i^{\frac{1}{l}}$. In the other case just denote $\mathrm{ex}(\frac{1}{l}a_i) = b_i^{\frac{1}{l}}$.
Choose now $\langle a_{k+1}, \ldots, a_n \rangle \in V_{k+1,\ldots,n}$, the variety induced by V on the last $n-k$ coordinates. By the above noted these are in R so we can define for $k < i \leq n$ $b_i^{\frac{1}{l}} = \mathrm{ex}(\frac{1}{l}a_i)$. Then $\langle b_{s+1}, \ldots, b_n \rangle$ will satisfy $W_{s+1,\ldots,n}$ since the last $n-k$ coordinates are algebraically independent over the preceding ones. In case the kernel is compact, by the choice of $\langle b_{s+1}, \ldots, b_k \rangle^{\frac{1}{l}}$ this is also true for $W^{\frac{1}{l}}_{s+1,\ldots,n}$.
Thus $\langle a_{s+1}, \ldots, a_n \rangle$ realises the type $\tau_{s+1,\ldots,n}$ induced by τ on the last $n-s$ coordinates.
Finally notice that the type $\tau(a_{s+1}, \ldots, a_n)$ over $C \cup \{a_{s+1}, \ldots, a_n\}$, corresponding to the first s coordinates in τ, satisfies the assumptions of Lemma 3.8. Thus it has a realisation $\langle a_1, \ldots, a_s \rangle$ in $\mathbf{D}$. Which completes the construction of $\langle a_1, \ldots, a_n \rangle$. $\Box$

Combined with Lemma 3.6 we thus get

Corollary 3 *(i) The set of incomplete ex-loci of $\bar{a}$ over $C \leq \mathbf{D} \in \mathcal{SC}$ for $\bar{a}$ living in $\mathbf{D}$ does not depend on $\mathbf{D}$.*
(ii) If the kernel K is compact then the set of (complete) ex-loci of $\bar{a}$ over $C \leq \mathbf{D} \in \mathcal{SC}$ for $\bar{a}$ living in $\mathbf{D}$ does not depend on $\mathbf{D}$.

Theorem 1 *A structure $\mathbf{D} \in \mathcal{S}(K)$ is in $\mathcal{SC}(K)$ iff for any irreducible normal free pair (V, W) over $\mathbf{D}$ there is a realisation of the type (V, W) in $\mathbf{D}$.*

First we prove

Lemma 3.10 *For any irreducible free normal pair (V, W) ($V \subseteq D^n, W \subseteq R^n$) and $V' \subseteq V$, $W' \subseteq W$ there is a free normal pair (V^*, W^*) ($V^* \subseteq D^{n+m}, W \subseteq R^{n+m}$) such that $\langle a_1, \ldots, a_n \rangle \in D^n$ realises $(V \setminus V', W \setminus W')$ iff there is $\langle a_{n+1}, \ldots, a_{n+m} \rangle \in D^m$ such that $\langle a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m} \rangle$ realises (V^*, W^*).*

Let $f(x_1, \ldots, x_n)$ be a polynomial in $\mathrm{Ann}V' \setminus \mathrm{Ann}V$. Choose a natural number k such that there is no rational function r such that $r^k \equiv f$ on V. Add new variable x_{n+1} together with new identity $f(x_1, \ldots, x_n) \cdot x_{n+1}^k = 1$, which defines together with $\mathrm{Ann}V$ new algebraic variety V^f. Notice that the projection of V^f onto the first n coordinates is $V \setminus V_f$ where V_f is the subvariety of V defined by $f = 0$. Also, V^* is free of additive dependencies since otherwise $\sqrt[k]{f} = x_{n+1}$ is a rational function on V. Using this method finitely many times we come to $V_1^* \subseteq D^{n+l}$ with the projection $V \setminus V'$.
Let W_1 be just $W \times D^l$. By construction (V_1^*, W_1^*) satisfies the statement of the lemma for (V^*, W^*) in case $W' = \emptyset$.

Assume now $W' \neq \emptyset$. Let $g(y_1, \ldots, y_n)$ be a polynomial in $\mathrm{Ann}W' \setminus \mathrm{Ann}W$. We may assume that there is no integer k and $m_1, \ldots, m_n$ not all zero such that

$$g(y_1, \ldots, y_n)^k \cdot y_1^{m_1} \cdot \ldots \cdot y_n^{m_n}$$

is constant on W since otherwise $g(y_1, \ldots, y_n)$ is nonvanishing on W. Add new variable y_{n+1} together with new identity $g(y_1, \ldots, y_n) \cdot y_{n+1} = 1$, and the resulting variety W_1^g projects onto the first n coordinates as $W \setminus W_g$ where W_g is the subvariety of W defined by $g = 0$. By our assumption W_1^g is free of multiplicative dependencies and we can come to W^* as above.
The normality of the pair is evident from the construction.$\square$

Proof of Theorem. The left-to-right implication follows from Lemma 3.8. To get the inverse assume $\bar{a}$ is in some $\mathbf{D}' \geq \mathbf{D}$ and we need to realise an almost finite type $(V \setminus V', W \setminus W', \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$ where $\tau = (V, W, \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$

is the ex-locus of $\bar{a}$ over D. It is enough to solve the problem for a $\mathbf{Q}$-linear basis $\bar{a}_0$ of $\bar{a}$ over D, so we may assume $\bar{a}$ is $\mathbf{Q}$-linearly independent over D. Thus (V, W) is a free normal pair, and so we may assume τ is a finite type. By Lemma 3.10 we reduce the type $(V \setminus V', W \setminus W')$ to a type of the form (V, W) and (V, W) normal and free. By the assumptions of Theorem the type is realised in $\mathbf{D}$. $\square$

Corollary 4 *The structure* $(\mathbf{C}, \exp, \mathbf{C})$ *on complex numbers is in* $\mathcal{SC}$ *iff it satisfies the weak Schanuel conjecture and for any normal free pair* (V, W) *over* $\mathbf{C}$ *there is a realisation of the type* (V, W) *in* $\mathbf{C}$.

4 Atomic and prime structures

In this section $\mathbf{D}$ is always exponentially-algebraically closed with a fixed kernel, the sort R is of characteristic zero. We consider two cases: the standard kernel $\pi\mathbf{Z}$ and the canonical compact kernel $\pi\hat{\mathbf{Z}}$. **Definition** Let $C_0 \subseteq C_1 \subseteq C \leq D$ and $V, V' \subseteq D^n$ algebraic varieties over C_0, $W, W' \subseteq R^n$ varieties over $\mathrm{ex}(C_0)$, V and W irreducible over corresponding sets and an associated sequence $\{W^{\frac{1}{l}} : l \in \mathbf{N}\}$ defined over C_1. The type $\tau = (V \setminus V', W \setminus W', \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$ is said to be a **q-atom over** C **weakly defined over** C_0, **defined over** C_1, if there is a realisation $\bar{a}$ of τ in some $\mathbf{D}' \geq \mathbf{D}$ and for every $\bar{a}$ realising τ in a $\mathbf{D}' \geq \mathbf{D}$, $\bar{a}$ **is generic over** C **realisation of** τ, i.e. $\bar{a}$ generic in V over C and $\mathrm{ex}(\bar{a})$ generic in W over $\mathrm{ex}(C)$.

C_0 is said to be weakly isolating $\bar{a}$ over C and C_1 isolating $\bar{a}$ over C.
Any realisation $\bar{a}$ in $\mathbf{D}$ of a type as above is said to be **q-atomic over** C.

Remark It follows immediately from definitions that for $\bar{a}$ and τ as above q-atomic over C, $\mathrm{ex}(\frac{1}{l}\bar{a})$ is also generic in $W^{\frac{1}{l}}$ for any $l \in \mathbf{N}$.

Remark It follows easily from definitions that if $\bar{a}$ and $\bar{a}'$ satisfy same q-atomic type over C, then the substructures $\mathrm{span}(C \cup |\bar{a}|)$ and $\mathrm{span}(C \cup |\bar{a}'|)$ are isomorphic.

Notation For $C \subseteq D$, $\mathbf{D} \in \mathcal{SC}$, denote $\mathrm{ecl}_D(C)$ the field obtained by the following process in ω steps carried on inside $\mathbf{D}$:
$C^0 = \mathrm{acl}(C)$, $C^{(2n+1)} = \ln(\mathrm{acl}(\mathrm{ex}(C^{(2n)})))$, $C^{(2n+2)} = \mathrm{acl}(C^{(2n+1)})$,

$$\mathrm{ecl}_D(C) = \bigcup_{n<\omega} C^{(n)}.$$

We omit the subscript when $\mathbf{D}$ is fixed.

Remark. If $X \subseteq C^{(m)}$, then by the definition

$$\delta(X/C^{(m-1)}) \leq 0$$

(here $C^{(-1)} = C$). If $C^{(m-1)} \leq D$, then the equality holds.

Lemma 4.1 *(i) For any $n \in \mathbf{N}$, $C \leq D$, $X \subseteq C^{(n)}$ there is $X \subseteq X' \subseteq C^{(n)}$, such that $\delta(X'/C) = 0$;*
(ii) If $C \leq D$, then $C^{(n)} \leq D$ for all $n \in \mathbf{N}$ and $\mathrm{ecl}(C) \leq D$;
(iii) If $X \subseteq \mathrm{ecl}(C)$ and $C \leq CX \leq D$, then $\delta(X/C) = 0$.

Proof (i) The statement is obvious for $n = 0$. So, we assume $n > 0$. Let $X \subseteq C^{(n)}$ and $Y \subseteq C^{(n-1)}$ be such that

$$\delta(X/C^{(n-1)}) = \delta(X/CY) = 0.$$

By induction hypothesis we may assume $\delta(Y/C) = 0$. Since

$$\delta(XY/C) = \delta(X/CY) + \delta(Y/C)$$

we get the required by putting $X' = XY$.
(ii) Let $X \subseteq D$ be finite. We want to show that $\delta(X/C^{(n)}) \geq 0$. By definition $\delta(X/C^{(n-1)}) = \delta(X/CY)$ for some finite $Y \subseteq C^{(n)}$. By (i) we may choose Y so that $\delta(Y/C) = 0$. Then $\delta(X/CY) = \delta(XY/C) \geq 0$, since $C \leq D$.
(iii) follows from (i). □

Lemma 4.2 *Let $C \leq D$, $|\bar{a}| \subseteq \mathrm{ecl}(C)$ and $C\bar{a} \leq D$. Then $\delta(\bar{a}/C) = 0$ and $\bar{a}$ is q-atomic over C.*

Proof $\delta(\bar{a}/C) = 0$ by Lemma 4.1(iii). To prove the rest of Lemma we first prove

Claim.

(i) If $|\bar{a}| \subseteq C^{(2n+2)}$, then there is $\bar{b}$ in $C^{(2n+1)} \cap \mathrm{span}_Q(C\bar{a})$, such that $\bar{a} \in \mathrm{acl}(C\bar{b})$ and $\delta(\bar{b}/C) = 0$.

(ii) If $|\bar{a}| \subseteq C^{(2n+1)}$, then there is $\bar{b}$ in $C^{(2n)} \cap \mathrm{span}_Q(C\bar{a})$, such that $\mathrm{ex}(\bar{a}) \in \mathrm{acl}(\mathrm{ex}(C\bar{b}))$ and $\delta(\bar{b}/C) = 0$.

Proof of (i). Choose finite $\bar{b}$ to be a $\mathbf{Q}$-basis of $C^{(2n+1)} \cap \mathrm{span}_Q(C\bar{a})$ Then $\dim_{\mathrm{Q}}(\bar{a}/C\bar{b}) = \dim_{\mathrm{Q}}(\bar{a}/C^{(2n+1)})$, since if a $\mathbf{Q}$-linear combination u of $\bar{a}$ belongs to $C^{(2n+1)}$, then $u \in C^{(2n+1)} \cap \mathrm{span}_Q(C\bar{a})$. It follows that $\delta(\bar{a}/C\bar{b}) = \mathrm{tr.d.}(\bar{a}/C\bar{b}) \geq 0$, since $\mathrm{tr.d.}(\mathrm{ex}(\bar{a})/\mathrm{ex}(C\bar{b})) \leq \dim_{\mathrm{Q}}(\bar{a}/C\bar{b})$ and
$\mathrm{tr.d.}(\mathrm{ex}(\bar{a})/\mathrm{ex}(C\bar{b})) \geq \mathrm{tr.d.}(\mathrm{ex}(\bar{a})/\mathrm{ex}(C^{(2n+1)})) = \dim_{\mathrm{Q}}(\bar{a}/C^{(2n+1)}) = \dim_{\mathrm{Q}}(\bar{a}/C\bar{b})$.
On the other hand, $\delta(\bar{a}/C\bar{b}) = \delta(\bar{a}\bar{b}/C) - \delta(\bar{b}/C) = 0 = \delta(\bar{b}/C)$, since $\delta(\bar{a}\bar{b}/C) = \delta(\bar{a}/C) = 0$ and $\delta(\bar{b}/C) \geq 0$ by $C \leq D$. Hence $\mathrm{tr.d.}(\bar{a}/C\bar{b}) = 0$, which means $|\bar{a}| \subseteq \mathrm{acl}(C\bar{b})$.

Proof of (ii) is symmetric.

Now we continue the proof of Lemma. By induction on m we prove that if $|\bar{a}| \subseteq C^{(m)}$, then $\bar{a}$ is q-atomic over C. For $m = 0$ $\bar{a} \in \mathrm{acl}(C)$ and the statement is evident. If $|\bar{a}| \subseteq C^{(m)}$ for $m > 1$, then by the Claim there is $\bar{b}$ in $C^{(m-1)} \cap \mathrm{span}_Q(C\bar{a})$ such that $\bar{a} \in \mathrm{acl}(C\bar{b})$ or $\mathrm{ex}(\bar{a}) \in \mathrm{acl}(\mathrm{ex}(C\bar{b}))$. Let $\bar{a} \in \mathrm{acl}(C\bar{b}) \subseteq (C\bar{b})^{(0)}$, more precisely for some $C_0 \subseteq C$ finite $\bar{a}$ is weakly isolated by $C_0\bar{b}$ over $C\bar{b}$, $\bar{b} \in \mathrm{span}_Q(C_0\bar{a})$, $C_0\bar{b} \leq D$ and C_0 weakly isolates $\bar{b}$ over C. Let $\tau = (V, W, \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$ be a quantifier-free type over C_0 satisfied by $\bar{a}\bar{b}$, such that

$$\tau(\bar{b}) = (V(\bar{b}), W(\mathrm{ex}(\bar{b})), \{W^{\frac{1}{l}}(\mathrm{ex}(\bar{b})) : l \in \mathbf{N}\})$$

is a q-atom over $C\bar{b}$ and τ implies the q-atomic type τ_0 of $\bar{b}$ over C. Then $\bar{x}$ realises $\tau(\bar{b})$ implies $\bar{x} \in \mathrm{acl}(C_0\bar{b})$. Since $\bar{b} \in \mathrm{span}_Q(C_0\bar{a})$, there is a $\mathbf{Q}$-linear combination with parameters in C_0 $\bar{q} \cdot \bar{x}$, such that $\bar{b} = \bar{q} \cdot \bar{a}$. Then the type saying that $\bar{x}\bar{y}$ satisfies τ and $\bar{y} = \bar{q} \cdot \bar{x}$ is a q-atom of $\bar{a}$ over C. Indeed, if $\bar{a}'$ satisfies the type and $\bar{b}' = \bar{q} \cdot \bar{a}'$, then $\bar{a}'\bar{b}'$ satisfies τ. Then $\bar{b}'$ realises τ_0, thus $\bar{b}'$ is generic in τ_0 over C. It follows that $C\bar{b}'$ is isomorphic to $C\bar{b}$. Then $\tau(\bar{b}')$ is a q-atom over $C\bar{b}'$ satisfied by $\bar{a}'$. It implies that $\bar{a}'$ is generic in $\tau(\bar{b}')$ over $C\bar{b}'$, hence $\bar{a}'\bar{b}'$ is generic in τ over C. $\square$

Lemma 4.3 *If $C \leq D$, then*
For any $X \subseteq C^{(2n+2)}$

$$\text{tr.d.}(\text{ex}(X)/\text{ex}(C^{(2n+1)})) = \dim_{\mathbb{Q}}(X/C^{(2n+1)});$$

For any $X \subseteq C^{(2n+1)}$

$$\text{tr.d.}(X/C^{(2n)}) = \dim_{\mathbb{Q}}(X/C^{(2n)}).$$

Proof By definition of δ, using Lemma 4.1(ii).□

Lemma 4.4 *Assume $C \leq A \leq \mathbf{D}_1$ and $C \leq B \leq \mathbf{D}_2$ are isomorphic over C as substructures by an isomorphism $\varphi : A \to B$, $A = C + \text{span}(|\bar{a}|$ for a finite string $\bar{a}$, $C = \text{ecl}(C)$ is countable and the kernel in $\mathbf{D}_1, \mathbf{D}_2$ is standard. Then φ can be extended to an isomorphism $\varphi : \text{ecl}(A) \to \text{ecl}(B)$.*

Proof Claim 1. If φ extends to $A \to B$ then it extends to $\text{acl}(A) \to \text{acl}(B)$. For this take any φ preserving the field structure and notice that for any $\mathbf{Q}$-linearly independent over A elements $a_1, \ldots, a_n \in \text{acl}(A)$ the algebraic type of $\text{ex}(a_1), \ldots, \text{ex}(a_n)$ over $\text{ex}(A)$ is uniquely determined by the requirement of algebraic independence.

Claim 2. φ extends to $A + \text{span}(|\bar{x}|)$ for any $\bar{x}$ in $\ln(\text{acl}(\text{ex}(A)))$. Indeed, let $\text{ex}(\bar{x}))$ be in $\text{acl}(\text{ex}(A))$. Let $W_A^{\frac{1}{l}}$ for $l \in \mathbf{N}$ be the algebraic loci of $\text{ex}(\frac{1}{l} \cdot \bar{v})$ over $\text{ex}(A)$. W_A can be represented as $W(\text{ex}(\bar{a}))$ for some W, an irreducible variety over $\text{ex}(C)$, the latter algebraically closed. We may assume $\bar{x}\bar{a}$ linearly independent over C. By Lemma 3.3 the sequence $\{W_A^{\frac{1}{l}} : l \in \mathbf{N}\}$ is determined by $W_A^{\frac{1}{l}}$ for some $l \in \mathbf{N}$. Take $\bar{y}$ in $\mathbf{D}$ such that

$$\text{ex}(\frac{1}{l} \cdot \bar{y}) \in W_B^{\frac{1}{l}}$$

where $W_B^{\frac{1}{l}}$ is obtained by applying φ to $W_A^{\frac{1}{l}}$. Since $A, B \leq D$ and

$$\text{tr.d.}(\text{ex}(\bar{x})/\text{ex}(A)) = 0 = \text{tr.d.}(\text{ex}(\bar{y})/\text{ex}(B)),$$

necessarily

$$\text{tr.d.}(\bar{x}/A) = n = \text{tr.d.}(\bar{y}/B),$$

i.e. both strings are algebraically independent over the corresponding sets. Hence one can extend $\varphi(\bar{x}) = \bar{y}$.
Claim 3. If $x \in \text{ecl}(A)$, then there is $\bar{x}$ in $\text{ecl}(A)$ extending x, such that $A\bar{x} \leq D_1$ and there is an isomorphism $A + \text{span}(|\bar{x}|) \to \text{ecl}(B)$ extending φ.
Proof of Claim 3. Take for $\bar{x}$ a finite string from $\text{ecl}(A)$ extending x such that $A\bar{x} \leq D$. By Lemma 4.2 $\bar{x}$ up to the ordering can be represented as $\bar{x}_0 \dots \bar{x}_k$ such that $\bar{x}_{i+1}$ is in $\text{acl}(A\bar{x}_0 \dots \bar{x}_i)$ or $\text{ex}(\bar{x}_{i+1})$ is in $\text{acl}(\text{ex}(A\bar{x}_0 \dots \bar{x}_i))$ depending on i odd or even.

Since C is countable, $\text{ecl}(A)$ and $\text{ecl}(B)$ can be enumerated and by back-and-forth arguments using Claim 3 one gets the isomorphism. $\square$

Corollary 5 *Suppose* $\mathbf{D}_1, \mathbf{D}_2$ *are with standard kernel,* $A \leq D_1$ *and* $B \leq D_2$ *are* ∂*-independent sets of cardinality* $\lambda \leq \omega_1$, $\varphi : A \to B$ *is a bijection. Then* φ *can be extended to an isomorphism* $\varphi : \text{ecl}(A) \to \text{ecl}(B)$.

Proof First enumerate A and B by ordinals agreeing with φ. We can then define for each $i < \lambda$ $A_i \leq A$ such that

$$A_{i+1} = \{a_j : j < i\}),$$

and correspondingly B_i.
Put $\varphi_0 = \varphi_{|\text{ecl}(\emptyset)}$ and proceed by induction constructing an isomorphism

$$\varphi_i : \text{ecl}(A_i) \to \text{ecl}(B_i).$$

Indeed, when a countable $\text{ecl}(A_i)$ is extended by an element we can apply Lemma and back-and-forth arguments to construct φ_{i+1}. Notice that, since A is ∂-independent, $\varphi \cup \varphi_i$ is still an isomorphism.
Take unions on limit steps.$\square$

Proposition 3 *Assume $A \leq \mathbf{D}_1$ and $B \leq \mathbf{D}_2$ are isomorphic as substructures by an isomorphism $\varphi : A \to B$ and the kernel in $\mathbf{D}_1, \mathbf{D}_2$ is compact. Then φ can be extended to an isomorphism $\varphi : \mathrm{ecl}(A) \to \mathrm{ecl}(B)$.*

Proof First we show that for any $x \in \ln(\mathrm{acl}(\mathrm{ex}(A)))$ φ extends to $A + \mathrm{span}(x) \to \mathbf{D}_2$.
Indeed, let $W_A^{\frac{1}{l}}$ for $l \in \mathbf{N}$ be the algebraic loci of $\mathrm{ex}(\frac{1}{l} \cdot x)$ over $\mathrm{ex}(A)$. By Lemma 3.2 there is $y \in D_2$ such that

$$\mathrm{ex}(\frac{1}{l} \cdot y) \in W_B^{\frac{1}{l}}$$

for all $l \in \mathbf{N}$, where $W_B^{\frac{1}{l}}$ are obtained by applying φ to $W_A^{\frac{1}{l}}$. Since $A \leq \mathbf{D}_1$, $B \leq \mathbf{D}_2$ and

$$\mathrm{tr.d.}(\mathrm{ex}(x)/\mathrm{ex}(A)) = 0 = \mathrm{tr.d.}(\mathrm{ex}(y)/\mathrm{ex}(B)),$$

necessarily

$$\mathrm{tr.d.}(x/A) = 1 = \mathrm{tr.d.}(y/B),$$

i.e. both elements are algebraically independent over the corresponding sets. Hence one can define

$$\varphi(a + \frac{1}{l} \cdot x) = b + \frac{1}{l} \cdot y$$

for any $a \in A$ and $b = \varphi(a)$.
Now notice that Claim 1 of previous Lemma holds independently on the kernel and the cardinality of A. Combining it with the above proved we can construct the required extension of φ by transfinite induction and back-and-forth argument.$\square$

Lemma 4.5 *Let (V, W) be a free normal pair in $\mathbf{D}^n$ over some $C_0 \leq \mathbf{D}$ such that $\dim V + \dim W - n = d > 0$ and $\dim V > 1$. Then for any $c_1, \ldots, c_n \in D$ algebraically independent over C_0 there is $V' \subseteq V$ defined over $C_0\bar{c}$ ($\bar{c} = \langle c_1, \ldots, c_n \rangle$) such that (V', W) is a normal free pair and $\dim V' + \dim W - n = d - 1$. If $\bar{a}$ realises (V', W) then $\delta(\bar{c}/C_0\bar{a}) < n$.*

Proof Let V' be an irreducible over $C_0\bar{c}$ component of the variety

$$V \cap \{\bar{x} : c_1 \cdot x_1 + \ldots + c_n \cdot x_n = 1\}.$$

Let $\bar{a}$ be a generic over $C_0\bar{c}$ point in V'. Then $\bar{a}$ is generic in V over C_0 and tr.d.$(\bar{a}/C_0\bar{c}) = \dim V - 1$, tr.d.$(\bar{c}/C_0\bar{a}) = n - 1$. In particular, it follows that $\delta(\bar{c}/C_0\bar{a}) < n$.

Claim 1. For any $k \leq n$ and any distinct $i_1, \ldots, i_k \in \{1, \ldots, n\}$ tr.d.$(a_{i_1}, \ldots, a_{i_k}/C_0) >$ tr.d.$(a_{i_1}, \ldots, a_{i_k}/C_0\bar{c})$ implies tr.d.$(a_{i_1}, \ldots, a_{i_k}/C_0) =$ tr.d.$(\bar{a}/C_0)$.

Proof Let $U \subseteq R^n$ be the minimal algebraic variety over $C_0 \cup \{a_{i_1}, \ldots, a_{i_k}\}$ containing $\bar{c}$. Suppose the inequality holds. Then $\dim U < n$. Also, U must contain the hyperplane H_a defined by $\bar{z}\bar{a} = 1$, for otherwise tr.d.$(\bar{c}/C_0\bar{a}) < n - 1$. Hence H_a is a component of U. But then $\bar{a} \in \mathrm{acl}(C_0 \cup \{a_{i_1}, \ldots, a_{i_k}\})$, since $\bar{a}$ is the only vector satisfying $\bar{z}\bar{a} = 1$ for all $\bar{z} \in H_a$. It follows that tr.d.$(a_{i_1}, \ldots, a_{i_k}/C_0) =$ tr.d.$(\bar{a}/C_0)$.

Claim 2. For any non-zero integer n-tuple $\bar{m} = \langle m_1, \ldots, m_n \rangle$ the element $\bar{m} \cdot \bar{a} = m_1 \cdot a_1 + \ldots + m_n \cdot a_n$ is not in $\mathrm{acl}(C_0\bar{c})$.

Proof Notice first that $a \notin \mathrm{acl}(C_0)$, since V is free of additive dependencies. Thus, if $a \in \mathrm{acl}(C_0\bar{c})$, then tr.d.$(\bar{c}/C_0a) < n$. By the argument from the proof of Claim 1 it would follow that $|\bar{a}| \subseteq \mathrm{acl}(C_0a)$. This contradicts the fact that $\dim V > 1$.

Proof of Lemma. It follows from Claim 2, that V' is free of additive dependencies. Thus the pair (V', W) is free. To prove the normality of the pair consider for any distinct $i_1, \ldots, i_k \in \{1, \ldots, n\}$ the number

$$\dim V'_{i_1,\ldots,i_k} + \dim W_{i_1,\ldots,i_k} - k.$$

Suppose towards the contradiction that the number is negative. Then

$$\dim V_{i_1,\ldots,i_k} + \dim W_{i_1,\ldots,i_k} - k \leq 0,$$

$\dim V'_{i_1,\ldots,i_k} \geq \dim V_{i_1,\ldots,i_k} - 1$. It follows that $\dim V'_{i_1,\ldots,i_k} < \dim V$, since $\dim W \leq \dim W_{i_1,\ldots,i_k} + (n-k)$ and $d > 0$. Thus

$$\text{tr.d.}(a_{i_1}, \ldots, a_{i_k}/C_0) < \text{tr.d.}(\bar{a}/C_0).$$

Then by Claim 1

$$\text{tr.d.}(a_{i_1}, \ldots, a_{i_k}/C_0) = \text{tr.d.}(a_{i_1}, \ldots, a_{i_k}/C_0\bar{c}).$$

Thus $\dim V'_{i_1,\ldots,i_k} = \dim V_{i_1,\ldots,i_k}$ and hence

$$\dim V_{i_1,\ldots,i_k} + \dim W_{i_1,\ldots,i_k} - k = \dim V'_{i_1,\ldots,i_k} + \dim W_{i_1,\ldots,i_k} - k < 0,$$

contradicting the normality of (V, W).
To finish the proof of normality we need to consider also

$$a'_i = m_{i,1}a_1 + \ldots + m_{i,n}a_n, \quad \text{and } b'_i = b_1^{m_{i,1}} \cdot \ldots \cdot b_n^{m_{i,n}} \quad i = 1, \ldots n$$

where $\langle m_{i,1}, \ldots, m_{i,n}\rangle$ are independent integer vectors and $\langle b_1, \ldots, b_n\rangle$ a generic element of W over $\mathrm{ex}(C_0\bar{c})$. The corresponding inequalities for the loci over $C_0\bar{c}$ and $\mathrm{ex}(C_0\bar{c})$ of the $\bar{a}'$ and $\bar{b}'$ are proved by the same argument. □

Lemma 4.6 *Let $C \leq D$ and $V, V' \subseteq D^n$ algebraic varieties over a finite $C_0 \subseteq C$, $W, W' \subseteq R^n$ varieties over $\mathrm{ex}(C_0)$. Let $\bar{a}$ in $\mathbf{D}$ realise the type $\tau = (V \setminus V', W \setminus W', \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$ and satisfy the following minimality condition:*
for any $\bar{a}' \in \mathbf{D}$ realising τ either $\dim_{\mathrm{Q}}(\bar{a}'/C) > \dim_{\mathrm{Q}}(\bar{a}/C)$ or $\dim_{\mathrm{Q}}(\bar{a}'/C) = \dim_{\mathrm{Q}}(\bar{a}/C)$ and $\delta(\bar{a}'/C) \geq \delta(\bar{a}/C)$.
Suppose the ex-locus $(V_0, W_0, \{W_0^{\frac{1}{l}} : l \in \mathbf{N}\})$ of $\bar{a}$ over C is weakly defined over C_0. Then $\tau_0 = (V_0 \setminus V', W_0 \setminus W', \{W_0^{\frac{1}{l}} : l \in \mathbf{N}\})$ is a q-atom over C weakly defined over C_0. If $C = \mathrm{ecl}(C)$, then $\delta(\bar{a}/C) = 0$ and there is a finite $C_1 \subseteq C$ such that τ is defined over C_1.

Proof Let $\bar{a}'$ be any realisation of τ_0 and assume towards a contradiction that $\bar{a}'$ is not generic over C. Then

$$\mathrm{tr.d.}(\bar{a}'/C) + \mathrm{tr.d.}(\mathrm{ex}(\bar{a}')/\mathrm{ex}(C)) < \mathrm{tr.d.}(\bar{a}/C) + \mathrm{tr.d.}(\mathrm{ex}(\bar{a})/\mathrm{ex}(C)).$$

By definition $\dim_{\mathrm{Q}}(\bar{a}'/C) \geq \dim_{\mathrm{Q}}(\bar{a}/C)$, so $\delta(\bar{a}'/C) < \delta(\bar{a}/C)$, the contradiction. This proves the first part of the statement.
Assume $C = \mathrm{ecl}(C)$ and assume towards a contradiction that $\delta(\bar{a}/C) = d > 0$. We can reduce the problem to a $\mathbf{Q}$-linear basis of $\bar{a}$ over C, so we assume that $\bar{a} = \langle a_1, \ldots, a_n\rangle$ is $\mathbf{Q}$-linearly independent over C. It follows that (V_0, W_0) is absolutely free and normal.
By the assumptions $\dim V_0 + \dim W_0 - n = \delta(\bar{a}/C) > 0$. Take $c_1, \ldots, c_n \in C$ algebraically independent over C_0 (such ones exist in $C_0^{(1)}$). By Lemma 4.5

either $\dim V_0 = 1$, or there is $V_0' \subseteq V_0$ over $\bar{C}_0\bar{c}$ such that (V_0', W_0) is free and normal. In the second case by Lemma 3.8 there is a generic over $C_0\bar{c}$ realisation $\bar{a}'$ of

$$(V_0', W_0, \{W_0^{\frac{1}{l}} : l \in \mathbf{N}\}),$$

which is also generic realisation of τ_0 over C_0, such that

$$\delta(\bar{a}'/C_0\bar{c}) = \dim V_0' + \dim W_0 - n < d.$$

The contradiction. In the first case, since $d > 0$, necessarily $\dim W_0 = n$, i.e. $W_0 = R^n$. Choose $c \in C \setminus C_0^{(1)}$ and let

$$W_0' = \{\langle y_1, \ldots, y_n \rangle : y_1 + \ldots + y_n = c\}.$$

Since $\mathrm{ex}(c) \notin \mathrm{acl}(\mathrm{ex}(C_0))$, any generic over $\mathrm{ex}(C_0 c)$ realisation $\langle b_1, \ldots, b_n \rangle$ of W' is generic in W over $\mathrm{ex}(C_0)$. Also, if $n > 1$, for any $k < n$ and distinct $i_1, \ldots, i_k \in \{1, \ldots, n\}$

$$\mathrm{tr.d.}(b_{i_1}, \ldots, b_{i_k}/\mathrm{ex}(C_0 c)) = \mathrm{tr.d.}(b_{i_1}, \ldots, b_{i_k}/\mathrm{ex}(C_0)),$$

which yields the normality of (V_0, W_0'). The same argument shows that $b_1^{m_1} \cdot \ldots \cdot b_n^{m_n} \notin \mathrm{acl}(\mathrm{ex}(C_0 c))$ for any non-zero integer n-tuple $\langle m_1, \ldots, m_n \rangle$, which shows that W_0' is free of multiplicative dependencies. Thus (V_0, W_0') is free. Again it gives by Lemma 3.8 as above a generic over C_0 realisation of τ_0 over C_0, such that

$$\delta(\bar{a}'/C_0\bar{c}) = \dim V_0 + \dim W_0' - n < d.$$

The final contradiction.
Finally the associated sequence is defined over some finite C_1 by Corollary 1.
$\square$

Lemma 4.7 *Let* $\mathrm{ecl}(\emptyset) \subseteq C \leq D$, $C \setminus \mathrm{ecl}(\emptyset)$ *be finite,* $\bar{a}\bar{b}$ *a tuple in* $\mathbf{D}' \geq \mathbf{D}$, $(V \setminus V_1, W \setminus W_1, \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$ *be be a type over* C *realised by* $\bar{a}\bar{b}$, $(V_0, W_0, \{W_0^{\frac{1}{l}} : l \in \mathbf{N}\})$, (V, W) *the main part of ex-locus of the tuple, and assume* $\delta(\bar{a}/C) = 0$. *Then there is* (V_0', W_0') *over* C *such that for any* $\bar{a}'$ *realising* $\sigma = (V_0 \setminus V_0', W_0 \setminus W_0', \{W_0^{\frac{1}{l}} : l \in \mathbf{N}\})$ *there is* $\bar{b}'$ *in some* $\mathbf{D}'' \geq \mathbf{D}'$ *such that* $\bar{a}'\bar{b}'$ *realise* τ.

Proof We just let $V_0 \setminus V_0'$ be the projection of V onto the coordinate space corresponding to $\bar{a}$ and $W_0 \setminus W_0'$ the projection of W.
We prove the statement by induction on $|\bar{b}| = n \geq 1$ assuming also that $\bar{a}\bar{b}$ are $\mathbf{Q}$-linearly independent over C. It follows from the assumptions that (V, W) is an absolutely free and normal pair.
Let $n = 1$. If both $\dim V(\bar{a}) = 1$ and $\dim W(\mathrm{ex}(\bar{a})) = 1$, then, by the elementary algebraic geometry, for any $\bar{a}'$ realising σ $\dim V(\bar{a}') = 1$ and $\dim W(\mathrm{ex}(\bar{a}')) = 1$. Hence $V(\bar{a}')$ and $W(\mathrm{ex}(\bar{a}'))$ are just the whole affine lines and we can take for b' any point in D.
If $\dim V(\bar{a}) = 0$, then necessarily $\dim W(\mathrm{ex}(\bar{a})) = 1$ and again $\dim V(\bar{a}') \geq 0$ and $\dim W(\mathrm{ex}(\bar{a}')) = 1$. Take for b' any point in $V(\bar{a}')$.
If $\dim W(\mathrm{ex}(\bar{a})) = 0$, then $\dim V(\bar{a}) = 1$ and $\dim W(\mathrm{ex}(\bar{a}')) \geq 0$, $\dim V(\bar{a}') = 1$.
Let

$$\tau_a = (V(\bar{a}\bar{c}), W(\mathrm{ex}(\bar{a}\bar{c})), \{(W(\mathrm{ex}(\bar{a}\bar{c})))^{\frac{1}{l}} : l \in \mathbf{N}\})$$

be the quantifier-free type of $\bar{b}$ over $C\bar{a}$, where $\bar{c}$ is a finite string from C and V, W are irreducible varieties over $\mathrm{ecl}(\emptyset)$, $\mathrm{ex}(\mathrm{ecl}(\emptyset))$, correspondingly.
$\tau_{a'}$ is the type obtained by replacing all occurencies of $\bar{a}$ by $\bar{a}'$. By Lemma 3.3 the sequence $\{(W(\mathrm{ex}(\bar{a}'\bar{c})))^{\frac{1}{l}} : l \in \mathbf{N}\}$) is determined by its cut of a length l_0, thus there is y in R and an associated sequence such that

$$y^{\frac{1}{l}} \in (W(\mathrm{ex}(\bar{a}'\bar{c})))^{\frac{1}{l}} \text{ for all } l \in \mathbf{N}\}.$$

Take $b' \in D$ such that

$$\mathrm{ex}(\frac{1}{l_0} \cdot b') = y^{\frac{1}{l_0}}.$$

Then

$$\mathrm{ex}(\frac{1}{l} \cdot b') \in (W(\mathrm{ex}(\bar{a}'\bar{c})))^{\frac{1}{l}} \text{ for all } l \in \mathbf{N}.$$

Since $V(\bar{a}') = D$, b' satisfies $\tau_{a'}$ and $\bar{a}'b'$ satisfies τ.
For $n > 1$ notice first that by the assumptions $C\bar{a} \leq D$, thus $(V(\bar{a}), W(\mathrm{ex}(\bar{a})))$ is a normal pair. To prove the statement of Lemma we consider two alternative cases:
Case 1. There are distinct $i_1, \dots i_k$ in $\{1, \dots, n-1\}$ such that the projections $V_{i_1,\dots i_k}(\bar{a})$ and $W_{i_1,\dots i_k}(\mathrm{ex}(\bar{a}))$ onto corresponding coordinates satisfy

$$\dim V_{i_1,\dots i_k}(\bar{a}) + \dim W_{i_1,\dots i_k}(\mathrm{ex}(\bar{a})) = k.$$

Let for simplicity $\langle i_1, \dots i_k\rangle = \langle 1, \dots k\rangle$. Then $\delta(\langle b_1, \dots b_k\rangle / C\bar{a}) = 0$ and one can consider $\bar{a}_+ = \bar{a}\langle b_1, \dots b_k\rangle$ in place of $\bar{a}$ and $\bar{b}_- = \langle b_{k+1}, \dots b_n\rangle$ in place of $\bar{b}$. By induction hypothesis find type σ_+ over C which says about any $\bar{a}'_+$ that there is $\bar{b}'_-$ such that $\bar{a}'_+\bar{b}'_-$ satisfy τ.
Case 2.

$$\dim V_{i_1,\dots i_k}(\bar{a}) + \dim W_{i_1,\dots i_k}(\mathrm{ex}(\bar{a})) > k$$

for any distinct $i_1, \dots i_k \in \{1, \dots, n-1\}$.
If the pair $(V(\bar{a}), W(\mathrm{ex}(\bar{a}))$ is free then the statement follows from Lemma 3.8. Suppose $W(\mathrm{ex}(\bar{a}))$ has a multiplicative dependence:

$$y_1^{m_1} \cdot \ldots \cdot y_n^{m_n} = r \in \mathrm{acl}(C\bar{a})$$

for some $m_1, \dots m_n \in \mathbf{Z}$, $r \in R$. Notice that by definitions

$$\mathrm{ex}(b_1)^{m_1} \cdot \ldots \cdot \mathrm{ex}(b_n)^{m_n} = r$$

holds, and so

$$m_1 b_1 + \ldots \cdot m_n b_n = d$$

holds for some $d \in \mathbf{D}$ such that $\mathrm{ex}(d) = r$. Assume $m_n \neq 0$. Apply the induction hypothesis considering $\bar{a}d$ in place of $\bar{a}$ and $\langle b_1, \dots, b_{n-1}\rangle$ in place of $\bar{b}$. We have then that whenever $\bar{a}'d'$ satisfy the same type as $\bar{a}d$, there is $\langle b'_1, \dots, b'_{n-1}\rangle$ such that $\bar{a}'\langle d', b'_1, \dots, b'_{n-1}\rangle$ is of the same type as $\bar{a}\langle d, b_1, \dots, b_{n-1}\rangle$. On the other hand, considering $n = 1$, we proved that whenever $\bar{a}'$ satisfies the type of $\bar{a}$, there is d' such that $\bar{a}'d'$ satisfy the type of $\bar{a}d$, which completes the proof in the case.
If $V(\bar{a})$ has an additive dependence we act symmetrically. $\square$

Lemma 4.8 *Let* $\mathrm{ecl}(\emptyset) \subseteq C \leq D$, $\bar{a}$, $\bar{b}$ *in* $\mathbf{D}$, $\delta(\bar{a}/C) = 0$ *and* $\bar{a}$ *q-atomic over* C, $\bar{b}$ *q-atomic over* $C\bar{a}$. *Then* $\bar{a}\bar{b}$ *is q-atomic over* C.

Proof Choose finite $C_0 \subseteq C$ such that C_0 weakly isolates $\bar{a}$ over C and $C_0\bar{a}$ weakly isolates $\bar{b}$ over $C\bar{a}$. Let $\tau = (V \setminus V_1, W \setminus W_1, \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$ be the q-atom of $\bar{a}\bar{b}$ over C_0 and σ the type obtained by projecting the varieties onto $\bar{a}$-coordinates. Then σ is a q-atom weakly isolating $\bar{a}$ over C. Also, the type of $\bar{b}$ over $C\bar{a}$ is determined over $C_0\bar{a}$ by $\tau(\bar{a})$. We claim that whenever $\bar{a}'\bar{b}'$ in some $\mathbf{D}' \geq \mathbf{D}$ realises τ, $\bar{a}'\bar{b}'$ is generic over C. This would prove Lemma.

Assume $\bar{a}'\bar{b}'$ realises τ. Then $\bar{a}'$ satisfies σ and, since σ is a q-atom over C, the quantifier-free types of $\bar{a}$ and $\bar{a}'$ over C coincide.
Suppose towards a contradiction that $\bar{b}'$ is not generic in $\tau(\bar{a}')$ over C. Then $\bar{b}'$ satisfies $\tau(\bar{a}')$ and a pair $(V'(\bar{a}'), W'(\mathrm{ex}(\bar{a}')))$ with

$$\dim V'(\bar{a}') < \dim V(\bar{a}') \text{ or } \dim W'(\mathrm{ex}(\bar{a}')) < \dim W(\mathrm{ex}(\bar{a}')).$$

Since τ and (V', W') are defined over some finite set C_1, $C_0 \subseteq C_1 \subseteq C$, by Lemma 4.7 there is $\bar{b}''$ which satisfies $\tau(\bar{a})$ and $(V'(\bar{a}), W'(\mathrm{ex}(\bar{a}))$. This contradicts the assumption that $\tau(\bar{a})$ is a q-atom over C.

Lemma 4.9 *Let* $\mathrm{ecl}(\emptyset) \subseteq C \leq C\bar{a} \leq D$ *and* $\bar{a}\bar{b}$ *be q-atomic over* C. *Then* $\bar{a}$ *is q-atomic over* C *and* $\bar{b}$ *is q-atomic over* $C\bar{a}$.

Proof Let

$$\tau = (V \setminus V', W \setminus W', \{W^{\frac{1}{l}} : l \in \mathbf{N}\})$$

be the q-atomic type of $\bar{a}\bar{b}$ over C weakly defined over some finite C_0. Let $V_1 \setminus V_1', W_1 \setminus W_1'$ be the projections of $V \setminus V', W \setminus W'$ onto the coordinates corresponding to $\bar{a}, \mathrm{ex}(\bar{a})$ and

$$\sigma = (V_1 \setminus V_1', W_1 \setminus W_1', \{W_1^{\frac{1}{l}} : l \in \mathbf{N}\}).$$

Suppose $\bar{a}'$ realises type σ. Then, by Lemma 4.7, there is $\bar{b}'$ such that $\bar{a}'\bar{b}'$ realises τ. Since τ is a q-atom over C, $\bar{a}'\bar{b}'$ is generic in (V, W) over C. It follows $\bar{a}'$ is generic in (V_1, W_1) and $\bar{b}'$ generic in $(V(\bar{a}'), W(\mathrm{ex}(\bar{a}')))$. The first fact yields that $\bar{a}$ is q-atomic over C, the second fact, after assuming $\bar{a}' = \bar{a}$, yields $\bar{b}$ is q-atomic over $C\bar{a}$.□

Definition A structure $A \in sub\mathcal{S}$ is said to be **q-atomic over** $C \leq A$ if for any $\bar{a}$ in A such that $C\bar{a} \leq A$ the tuple $\bar{a}$ is q-atomic over C.

Lemma 4.10 *Assume* A *is with full kernel,* $\mathrm{ecl}(\emptyset) \subseteq C \leq A$.
(i) If $B \subseteq A$ *finite,* $\delta(B/C) = 0$ *and* A *is q-atomic over* C, *then* A *is q-atomic over* CB.
(ii) If $\mathrm{ecl}(C) \subseteq A$ *and* A *is q-atomic over* $\mathrm{ecl}(C)$, *then* A *is q-atomic over* C *and for any* $\bar{a}$ *in* A, *such that* $C\bar{a} \leq A$, *it holds* $\delta(\bar{a}/C) = 0$.

Proof (i) Suppose $CB\bar{a} \leq A$. By assumtions, $B\bar{a}$ is q-atomic over C. Then by Lemma 4.9 $\bar{a}$ is q-atomic over CB.
(ii) Suppose $C\bar{a} \leq A$. Extend $\bar{a}$ to $\bar{a}'$ so, that $\mathrm{ecl}(C)\bar{a}' \leq A$. Then, by assumtions, $\bar{a}'$ is q-atomic over $\mathrm{ecl}(C)$ and by 4.6 $\delta(\bar{a}'/\mathrm{ecl}(C)) = 0$.
Choose finite $\bar{c}$ in $\mathrm{ecl}(C)$ such that $\bar{a}'$ is q-atomic over $C\bar{c}$, $\delta(\bar{a}'/C\bar{c}) = 0$ and $C\bar{c} \leq A$. By Lemmas 4.2 and 4.8, $\bar{c}\bar{a}'$ is q-atomic over C. Also

$$\delta(\bar{c}\bar{a}'/C) = \delta(\bar{a}'/C\bar{c}) + \delta(\bar{c}/C) = 0.$$

Since $\bar{c}\bar{a}'$ extends $\bar{a}$ and $C\bar{a} \leq A$, $\delta(\bar{a}/C) = 0$. Finally, by Lemma 4.9, $\bar{a}$ is q-atomic over C. □

Proposition 4 *Over any $C \leq \mathbf{D}$ there is a q-atomic structure $E(C) \in \mathcal{SC}$, $E(C) \leq D$. If the kernel K is compact then for any embedding $C \leq \mathbf{D}' \in \mathcal{SC}$ there is an extension of the embedding $E(C) \leq \mathbf{D}'$.*
The cardinality of $E(C)$ is not greater than card (C) + card (K).

Proof $E_D(C) = E(C)$ will be represented as $\bigcup_n C_n$ for some

$$C_0 \leq \ldots \leq C_n \leq \ldots \subseteq \mathbf{D}.$$

Put $C_0 = \mathrm{ecl}(C)$ and assume C_n is constructed. Let $\{\mu_{n,\alpha} : \alpha < \lambda_n\}$ be the set of all types of the form (V, W), irreducible free and normal over C_n, $\lambda_n =$ card (C_n).
Put $C_{n,0} = C_n$ and, when $C_{n,\alpha}$ is constructed, use Lemma 4.6 to find in some normal extension of $C_{n,\alpha}$ a q-atomic over $C_{n,\alpha}$ tuple $\bar{a}_{n,\alpha}$ realising $\mu_{n,\alpha}$. Since, by Lemma 3.6 the incomplete ex-locus of $\bar{a}_{n,\alpha}$ over $\mathrm{ecl}_D(C_{n,\alpha})$ is absolutely free and normal, by Lemma 3.7 the corresponding type is realisable in $\mathbf{D}$, so we assume the tuple is in $\mathbf{D}$.
Put

$$C_{n,\alpha+1} = C_{n,\alpha} \cup |\bar{a}_{n,\alpha}|.$$

For limit ordinals σ

$$C_{n,\sigma} = \bigcup_{\alpha<\sigma} C_{n,\alpha}.$$

Finally, $C_{n+1} = C_{n,\lambda_n}$.
Claim. For any n and $\alpha < \lambda_n$

(i) $C_{n,\alpha}$ is q-atomic over C_0;
(ii) $C_{n,\alpha} \leq D$;
(iii) $C_{n,\alpha}$ is q-atomic over C;
(iv) if kernel is compact, $C_{n,\alpha}$ does not depend on $\mathbf{D}$.
Proof of Claim. By induction on lexicographically ordered pairs $\langle n, \alpha \rangle$.
For $\langle 0, 0 \rangle$ the statements follow from properties of ecl. The inductive step for limit ordinals is trivial, so we assume the Claim is true for $C_{n,\alpha}$ and want to prove it for $C_{n,\alpha+1}$.
Let $\bar{a}$ be a tuple from $C_{n,\alpha+1}$, such that $C_0\bar{a} \leq D$. By the construction there is $\bar{b}$ in $C_{n,\alpha}$ which weakly isolates $\bar{a}_{n,\alpha}$ over $C_{n,\alpha}$. When we extend $\bar{b}$ the property is preserved, so we assume $C_0\bar{b} \leq D$ and $|\bar{a}| \subseteq |\bar{b}\bar{a}_{n,\alpha}|$. Then, by induction, $\bar{b}$ is q-atomic over C_0 and by Lemma 4.6 $\delta(\bar{b}/C_0) = 0$. Now by Lemma 4.8 $\bar{b}\bar{a}_{n,\alpha}$ is q-atomic over C_0. Since $|\bar{b}\bar{a}_{n,\alpha}| = |\bar{a}\bar{b}'|$ for some $\bar{b}'$, by Lemma 4.9 $\bar{a}$ is q-atomic over C_0, which proves (i).
To prove (ii) we must show that $\delta(X/C_{n,\alpha+1}) \geq 0$ for any finite $X \subseteq D$. Notice that $\delta(X/C_{n,\alpha+1}) = \delta(X/C_0Y)$ for large enough finite $Y \subseteq C_{n,\alpha+1}$. We can choose $Y = |\bar{b}\bar{a}_{n,\alpha}|$ with $\bar{b}$ as above. Then Y is q-atomic over C_0 and by Lemma 4.6 $\delta(Y/C_0) = 0$. Hence $\delta(X/C_0Y) \geq 0$, by Lemma 2.4. Which finishes the proof of (ii).
(iii) follows from (i) and Lemma 4.10.
(iv) follows from Proposition 3.
It follows from the (i)-(iii) of the claim that $E(C)$ is q-atomic over C. It follows from Theorem1 that $E(C)$ is e.a.c. In case the kernel is compact it follows from (iv) of the claim that $E(C)$ can be embedded in any $\mathbf{D}'$ provided $C \leq \mathbf{D}'$.
The cardinality statement follows from the fact that $\lambda_n \leq$ card $(C_0) =$ card $(\text{ecl}(C) \leq$ card $(C) +$ card (K). $\square$

Definition A structure $E(C) \in \mathcal{SC}$ is said to be **constructible q-prime over** C if
(i) $C \leq E(C)$,
(ii) $E(C)$ is q-atomic over $\text{ecl}(C)$,
(iii) there is a sequence $\{\bar{a}_j : j < \lambda\}$ of tuples of $E(C)$ for some limit ordinal λ, such that $E(C) = \bigcup\{|\bar{a}_j| : j < \lambda\}$ and for any $j \leq \lambda$ $\bar{a}_j$ is a realisation of a q-atom τ_j over $\text{ecl}(C) \cup A_j$, where $A_j = \bigcup_{i<j} |\bar{a}_i|$. The corresponding sequence $\{(\bar{a}_j, \tau_j) : j < \lambda\}$ is said to be a **q-construction**. We assume

$|\bar{a}_j| \cap (\mathrm{ecl}(C) \cup A_j) = \emptyset$.

Proposition 5 *Suppose the kernel K is compact. Then over any $C \in sub\mathcal{S}$ there is a q-prime structure $E(C)$ such that for any embedding $C \leq \mathbf{D} \in \mathcal{SC}$ there is an extension of the embedding $E(C) \leq \mathbf{D}$. Any two constructible q-prime structures over C are isomorphic over C.*

Proof The first statement is proved in Proposition 4, where the sequence $\{\bar{a}_j : j < \lambda\}$ is just $\{\bar{a}_{n,\alpha} : n < \omega, \alpha < \lambda_n\}$ getting ordinal enumeration and τ_j are the corresponding types. To prove the rest we use the argument from [Sh] following [B].

We assume by 3 that $C = \mathrm{ecl}(C)$.
A subset B of a constructible q-prime structure over C is said to be closed if for all $\bar{a}_j$, if $|\bar{a}_j| \cap B \neq \emptyset$, then $|\bar{a}_j| \subseteq CB$, CB contains all parameters of type τ_j and $CB \leq E(C)$. Notice that each point $b \in B$ can be included in some finite string $\bar{b}$ from B such that $C\bar{b} \leq E(C)$. Since $E(C)$ is q-atomic, by Lemma 4.6 $\delta(\bar{b}/C) = 0$. In particular, if B is finite $\delta(B/C) = 0$.
Claim 1. If B is a closed subset of $E(C)$, $j < \lambda$, then
(i) $CBA_j \leq E(C)$;
(ii) $\delta(\bar{a}_j/CBA_j) = 0$;
(iii) $\bar{a}_j$ is q-atomic over CBA_j and the corresponding q-atom is τ_j.
Proof of Claim. For any $j < \lambda$ we now prove (i) - (iii) for $B^i = B \cap A_i$, in place of B for all $i \leq \lambda$ by induction on i.
For $i \leq j$ this is true, since then $B^i \subseteq A_j$, (i) and (iii) are true by the definition and (ii) follows by the following argument, which works for any i, provided (i) and (iii) hold. Choose finite $A \subseteq B^iA_j$, such that $CA \leq E(C)$ and A contains the parameters of τ_j. Then A is q-atomic over C and $\bar{a}_j$ is q-atomic over CA. By Lemma 4.9 $A\bar{a}_j$ is q-atomic over C, hence by Lemma 4.6 $\delta(A\bar{a}_j/C) = 0$, $\delta(A/C) = 0$. It follows $\delta(\bar{a}_j/CA) = 0$, which in turn implies $\delta(\bar{a}_j/CB^iA_j) = 0$.
Assume now that (i)-(iii) hold for some i, $j \leq i < \lambda$, we want to prove it for $i+1$. Assume $B^{i+1} \neq B^i$. Then $B^{i+1} = |\bar{a}_i| \cup B^i$. Since $B^iA_j \subseteq A_i$ and the parameters of τ_i are in B^i, $\bar{a}_i$ is q-atomic over $CB^iA_j\bar{a}_j$. By the induction hypothesis $\bar{a}_j$ is q-atomic over CB^iA_j by τ_j and $\delta(\bar{a}_j/CB^iA_j) = 0$. Then by

4.8 $\bar{a}_j\bar{a}_i$ is q-atomic over CB^iA_j. More precisely, the type of $\bar{a}_j\bar{a}_i$ over CB^iA_j is determined by $\tau_j(\bar{x})\&\tau_i(\bar{y})$ (where $\bar{x}$, $\bar{y}$ correspond to $\bar{a}_j$, $\bar{a}_i$).
Since $\bar{a}_i$ is q-atomic over CB^iA_j and $CB^iA_j \leq E(C)$, by the argument above $\delta(\bar{a}_i/CB^iA_j) = 0$. Then $CB^iA_j\bar{a}_i \leq E(C)$. Hence, by Lemma 4.9 and the above proved, $\bar{a}_i$ is q-atomic over CB^iA_j and $\bar{a}_j$ is q-atomic over $CB^iA_j\bar{a}_i$, more presicely the q-atom is again τ_j. Since $CB^iA_j\bar{a}_i = CB^{i+1}A_j$, we proved (i) and (iii) for B^{i+1}. Once again (ii) is proved by the argument above.
For limit ordinals i (i) - (iii) is evident. This finishes the proof of Claim.
Claim 2. Suppose B is closed. Then for any finite $X \subseteq E(C)$ there is finite $B_X \subseteq E(C)$, such that BB_X is closed and $X \subseteq B_X$.
Proof of Claim. We prove by induction on j that for $X \subseteq A_j$ there is $B_X \subseteq A_j$, such that BB_X is closed. If $j = 0$, $X = \emptyset = B_X$. Suppose $|\bar{a}_j| \subseteq X \subseteq A_{j+1}$. Choose finite $Y \subseteq A_j$, such that $X \cap BA_j \subseteq Y$, CBB_Y is closed and CB_Y contains all the parameters of τ_j. Then there is finite $Z \subseteq B$ such that $CZB_Y \leq CBB_Y \leq E(C)$ and $\bar{a}_j$ is q-atomic over CZB_Y. Since $E(C)$ is q-atomic over C, ZB_Y is q-atomic and by 4.6, 4.8 $ZB_y\bar{a}_j$ is q-atomic over C. Hence $\delta(ZB_y\bar{a}_j/C) = 0$, thus $CZB_Y\bar{a}_j \leq E(C)$. Since Z can be chosen as large as we want, it follows $CBB_Y \leq E(C)$. We state that $BB_Y\bar{a}_j$ is closed. It remains to prove that if there is $a \in |\bar{a}_i|$ such that $a \in BB_Y|\bar{a}_j|$ then $|\bar{a}_i| \subseteq CBB_Y|\bar{a}_j|$ and the parameters of τ_i are in the same set. By the construction it is true if $a \in BB_Y$. If $a \in |\bar{a}_j|$, then $i = j$ and both $|\bar{a}_j$ and the set of all parameters of τ_j are subsets of $CBB_Y|\bar{a}_j|$. Thus $BB_Y|\bar{a}_j|$ is a closed set which contains X. Claim proved.
Claim 3. $E(C)$ is q-atomic over CB for any closed B and if $CB\bar{a} \leq E(C)$, then $\delta(\bar{a}/CB) = 0$.
Proof of Claim. It is enough to prove that any $\bar{a}$ in CBA_j, such that $CB\bar{a} \leq E(C)$, $E(C)$ is q-atomic over CB and $\delta(\bar{a}/CB) = 0$ CB for all $j < \lambda$. We prove it by induction on j. The statement is trivial for $j = 0$. Suppose it holds for j and $\bar{a}$ is a tuple in CBA_{j+1} such that $CB\bar{a} \leq E(C)$. Choose finite tuple $\bar{b}$ in CBA_j such that $CB\bar{b} \leq E(C)$, $|\bar{a}| \subseteq |\bar{b}\bar{a}_j|$ and $\bar{a}_j$ is q-atomic over $C\bar{b}$ (Claim 1). By Lemma 4.8 $\bar{b}\bar{a}_j$ is q-atomic over CB. By Lemma 4.9 $\bar{a}$ is q-atomic over CB. Since $|\bar{b}\bar{a}_j| = |\bar{a}\bar{b}'|$ for some $\bar{b}'$, by Lemma 4.9 $\bar{a}$ is q-atomic over CB. For large enough $\bar{b}$

$$\delta(\bar{b}\bar{a}_j/CB) = \delta(\bar{a}_j/CB\bar{b}) + \delta(\bar{b}/CB) = \delta(\bar{a}_j/CB\bar{b}) = \delta(\bar{a}_j/CBA_j) = 0.$$

Since

$$0 = \delta(\bar{a}\bar{b}'/CB) = \delta(\bar{b}'/CB\bar{a}) + \delta(\bar{a}/CB)$$

and both summands on the right are non-negative by $CB \leq CB\bar{a} \leq E(C)$, we get $\delta(\bar{a}/CB) = 0$. Claim proved.

Now we come to the proof of Proposition. We consider two constructible q-prime structures A and A' over C with q-constructions $\{(\bar{a}_j, \tau_j) : j < \lambda\}$ and $\{(\bar{a}'_j, \tau'_j) : j < \lambda'\}$ correspondingly. Assume $\lambda \leq \lambda'$. We define by induction ascending chains of isomorphic closed sets $C_j \leq A$ and $C'_j \leq A'$ for $j < \lambda$ and the isomorphisms $f_j : C_j \to C'_j$. We also satisfy the condition that if $j = \eta + (2n+2)$ η limit, n natural, then $\bar{a}_{\eta+n}$ lives in C_j; if $j = \eta + (2n+1)$ then $\bar{a}'_{\eta+n}$ lives in C'_j.
Consider a typical case of the construction: $j = \eta + (2n+2)$. By Claim 2 there is a finite set B_0 containing $|\bar{a}_{\eta+n}|$, such that $C_{j-1}B_0$ is closed. By Claim 3 A is q-atomic over C_{j-1} and $\delta(B_0/C_{j-1}) = 0$. Since A is q-atomic over C_{j-1} there is an isomorphism g_0 extending f_{j-1} and taking B_0 onto some $B'_0 \subseteq A'$. By Claim 2 B'_0 can be included in some $B'_1 \subseteq A'$ such that $C'_{j-1} \cup B'_1$ is closed. By the isomorphism $\delta(B'_0/C'_{j-1}) = 0$, hence by Lemma 4.10 A' is q-atomic over $C'_{j-1}B'_0$. Thus B'_1 satisfies a q-atom over $C'_{j-1}B'_0$. It follows there is an isomorphism g_1 extending g_0 and taking $C_{j-1} \cup B_1$ for some B_1 onto $C'_{j-1} \cup B'_1$. Continuing in this way we get a chain of finite sets $B_m \subseteq A$ ($m \in \mathbf{N}$), $B'_m \subseteq A'$ and corresponding isomorphisms g_m from $C_{j-1}B_m$ onto $C'_{j-1}B'_m$, such that $C_{j-1}B_m$ is closed if m is even and $C'_{j-1}B'_m$ is closed if m is odd. Finally the sets

$$C_j = C_{j-1} \cup \bigcup_{m \in \mathbf{N}} B_m, \quad C'_j = C'_{j-1} \cup \bigcup_{m \in \mathbf{N}} B'_m \text{ and } f_j = \bigcup_{m \in \mathbf{N}} g_m$$

satisfy the conditions required for j. For odd j the procedure is symmetrically from A' to A. In the end the construction yields an isomorphism $f = f_\lambda$ from A onto the closed subset C'_λ of A'. Since A' is q-atomic over C'_λ and by the isomorphism any q-atom over C'_λ is realised in C'_λ we have necessarily $C'_\lambda = A'$.$\square$

Theorem 2 *Let $C \leq D$ and C be a ∂-independent set of power λ and $\kappa =$* card K. *Suppose that D is q-prime constructible over C. Then*
(i) given any C' maximal ∂-independent subset of A, card $(C) =$ card (C');
(ii) any bijection $g : C \to C$ is elementary, moreover, if the kernel is compact g can be extended to an automorphism;

(iii) for finite $S \subseteq D$ card $(\mathrm{cl}(S)) \leq \kappa$;
(iv) if $\lambda > \kappa$ then any definable subset of D is either of power λ or at most κ.

Proof (i) By Lemma 4.6 C' is in $\mathrm{cl}(C)$, the ∂-closure of C. By Lemma 3.5 card $(C') \leq$ card (C). On the other hand $\mathrm{cl}(C')$ must contain C for otherwise C' is not maximal. Hence card $(C') =$ card (C).

(ii) since g is a (partial) isomorphism we may identify elements of C with their images to apply Proposition 5. Indeed, we can see that q-construction $\{a_j : j < \lambda\}$ of $\mathbf{D}$ as a q-prime structure over C gives rise to the q-construction $\{a_j : j < \lambda\}$ over same C with types τ_j replaced by $g(\tau_j)$ obtained by exchanging parameters in correspondence with g. So by Proposition 5 g can be extended to an automorphism.
(iii) By the definitions and the above proved there is a finite subset $C_0 \subseteq C$, such that $S \subseteq \mathrm{cl}(C_0)$. Consider $E(C_0)$, which may be assumed embedded in $E(C)$. Now construct inside $E(C)$ structure $A = E(E(C_0) \cup C)$. Notice, that since $E(C_0) \subseteq \mathrm{cl}_A(C_0)$, the set $C \setminus C_0$ is independent over $E(C_0)$.
We now claim $\mathrm{cl}_A(C_0) = E(C_0)$. Let $a \in \mathrm{cl}_A(C_0)$. Since A is q-atomic over $E(C_0) \cup C$, there is $\bar{a}$ extending a, $\delta(\bar{a}/E(C_0) \cup C) = 0$ and such that $\bar{a}$ is q-atomic over $E(C_0) \cup C$. Let τ be the q-atom of $\bar{a}$ over the set. More precisely,

$$\tau = (V(\bar{c}), W(\mathrm{ex}(\bar{c})), \{W^{\frac{1}{l}}(\mathrm{ex}(\frac{1}{l}\bar{c})) : l \in \mathbf{N}\})$$

with $\bar{c}$ a finite string from $C \setminus C_0$ and (V, W) defined over a finite $B \subseteq E(C_0)$. On the other hand let

$$\tau_0 = (V_0, W_0, \{W_0^{\frac{1}{l}} : l \in \mathbf{N}\})$$

be the ex-locus ov $\bar{a}$ over $E(C_0)$. Since $\bar{c}$ is ∂-independent over $B\bar{a}$, $\dim_{\mathbb{Q}}(\bar{a}/B\bar{c}) = \dim_{\mathbb{Q}}(\bar{a}/B)$ and hence $\dim V(\bar{c}) = \dim V_0$, $\dim W(\mathrm{ex}(\bar{c})) = \dim W_0$. But V_0 and W_0, being defined over algebraically closed fields, are absolutely irreducible, hence $\tau = \tau_0$ is defined over B.
Since $E(C_0)$ is existentially-algebraically closed, τ is realised in $E(C_0)$ by a tuple $\bar{a}'$. Since τ is a q-atom over $E(C_0) \cup C$, we get $\bar{a} = \bar{a}'$, proving the claim. It follows from the claim and Proposition4 that card $(\mathrm{cl}_A(C_0) \leq$ card $(E(C_0)) \leq \kappa$. On the other hand $E(C)$ is embeddable in A and by this embedding $\mathrm{cl}(C_0)$ goes into $\mathrm{cl}_A(C_0)$. It finishes the proof of (iii).

(iv) Choose $C_0 \subseteq C$ finite such that $S \subseteq \mathrm{cl}(C_0)$. If a set X definable over finite $S \leq \mathbf{D}$ is of power greater than κ, then by (iii) it contains an element $b \notin \mathrm{cl}(C_0)$. But $b \in \mathrm{cl}(C)$, so we can find finite $C_1 \supseteq C_0$ and $c \in C$ such that $b \in \mathrm{cl}(C_1 c) \setminus \mathrm{cl}(C_1)$. Hence $\mathrm{cl}(C_1 b) = \mathrm{cl}(C_1 c)$. Consider bijections $g : C \to C$ which fix C_1. It follows that the automorphism of $\mathbf{D}$ induced by such g have the property that $g_1(b) \neq g_2(b)$ whenever $g_1(c) \neq g_2(c)$. Thus there are λ elements of the form $g(b)$, which all are in X. □

5 Pseudo-analytic dimension in canonical structures

Throughout this section $\mathbf{D}$ is a canonical structure over a ∂-independent set of power $\lambda > \aleph_0$ and the standard kernel. We expand the language by naming all elements of the standard kernel.

Definition For a subset $S \subseteq D^n$ definable over some finite $C \leq$ **D the pseudo-analytic dimension of** S, denoted adim S, is given as

$$\text{adim } S = \max_{\bar{a} \in S} \partial(\bar{a}/C).$$

Lemma 5.1 *(i) The definition of* adim *does not depend on the choice of parameters $C \leq D$.*
(ii) adim $S \geq 0$ *for any non-empty S.*
(iii) For non-empty S adim $S = 0$ *iff* card $S \leq \kappa$.
(iv) For non-empty S adim $S \geq m > 0$ *iff S can be projected to D^m so that the complement to the project ion in D^m is of pseudo-analytic dimension less than m or empty.*
(v) For algebraic varieties $V \subseteq D^n$, $W \subseteq R^n$

$$\text{adim } V = \dim V, \quad \text{adim } \ln W = \dim W.$$

Proof Immediate.□

Proposition 6 *Suppose* (V, W) *is a normal free pair over* $C \leq \mathbf{D}$. *Then*

$$\text{adim }(V \cap \ln W) \geq \dim V + \dim W - n.$$

Proof By induction on $d = \dim V + \dim W - n \geq 0$.
If $\dim W = n$, then $V \cap \ln W = V$. Thus we may assume $\dim V > 1$. Choose $c_1, \ldots, c_n \in D$ such that $\partial(\langle c_1, \ldots, c_n\rangle / C) = n$. By Lemma 4.5 we get then algebraic subvariety $V' \subseteq V$ defined over $C \cup \{c_1, \ldots, c_n\}$ such that pair (V', W) is normal, free and

$$\dim V' + \dim W' - n = d - 1.$$

By induction there is $\bar{a} \in V \cap \ln W$, $\partial(\bar{a}/C\bar{c}) = d - 1$. But, again by Lemma 4.5, $\partial(\bar{c}/C\bar{a}) < n = \partial(\bar{c}/C)$, hence $\partial(\bar{a}/C) > \partial(\bar{a}/C\bar{c}) = d - 1$. $\square$

Lemma 5.2 *Let* (V, W) *be a normal pair over* $C \leq \mathbf{D}$ *and suppose*

$$V \cap \ln W \neq \emptyset.$$

Then

$$\text{adim }(V \cap \ln W) \geq \dim V + \dim W - n.$$

Proof We use induction on n. The case $n = 1$ is trivial.
Consider the general case. Choose $\langle a_1, \ldots, a_n\rangle \in V \cap \ln W$. By the induction hypothesis there is $\langle a'_1, \ldots, a'_{n-1}\rangle \in V(a_n) \cap \ln W(\text{ex}(a_n))$ with

$$\partial(\langle a'_1, \ldots, a'_{n-1}\rangle / Ca_n) \geq \dim V(a_n) + \dim W(\text{ex}(a_n)) - (n-1).$$

If $\dim V(a_n) = \dim V$ or $\dim W(\text{ex}(a_n)) = \dim W$ then we would have the desired estimate

$$\partial(\langle a'_1, \ldots, a'_{n-1}\rangle / Ca_n) \geq \dim V + \dim W - n.$$

Hence we assume $\dim V(a_n) = \dim V - 1$ and $\dim W(\text{ex}(a_n)) = \dim W - 1$ which yields that the estimates are true for generic fibers and thus the projections V_n of V and W_n of W to the last coordinate are dense in D and R correspondingly. Choose $a'_n \in D$ ∂-independent over C.

By the induction hypothesis there is $\langle a'_1, \ldots, a'_{n-1}\rangle \in V(a'_n) \cap \ln W(\mathrm{ex}(a'_n))$ with

$$\partial(\langle a'_1, \ldots, a'_{n-1}\rangle / Ca'_n) \geq \dim V(a_n) + \dim W(\mathrm{ex}(a_n)) - (n-1) = \dim V + \dim W - n - 1.$$

By our choice

$$\partial(\langle a'_1, \ldots, a'_n\rangle / C) = \partial(\langle a'_1, \ldots, a'_{n-1}\rangle / Ca'_n) + 1 \geq \dim V + \dim W - n$$

and $\langle a'_1, \ldots, a'_n\rangle \in V \cap \ln W$. $\square$

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